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Laplace-variational method for transient multidimensional temperature distributions

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LAPLACE-VARIATIONAL METHOD FOR TRANSIENT
MULTIDIMENSIONAL TEMPERATURE DISTRIBUTIONS

by

Philip Arthur Loretan

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Nuclear Engineering

Approved:

Signature was redacted for privacy.

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Iowa State University
Of Science and Technology
Ames, Iowa

1965

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ABSTRACT

An analytical method has been developed which will provide closed form approximate temperature distributions for two-dimensional transient conduction heat transfer problems. It is referred to as a Laplace-variational method since it utilizes a Laplace transformation along with methods from the calculus of variations.

The Laplace-variational method can be applied to bodies with or without heat generation. Application of the method to both of these types of two-dimensional problems is shown in this dissertation. Also, some one-dimensional and three-dimensional transient conduction heat transfer problems can be analyzed by the Laplace-variational method even though the method was formally prepared for two-dimensional problems. The procedure used to approach such problems by this method is outlined. Of course, no matter how complex the structure, accuracy of the computed temperature distribution will normally be sacrificed somewhat when the Laplace-variational method is utilized since an approximate solution results from the analysis. The accuracy of the solution can sometimes be improved but only at an increase in cost.

Use of the Laplace-variational method in heat transfer problems means that the thermal properties of a material must be constant and also that a radiation boundary condition cannot be considered. Utilization of the technique in this

thesis has also been restricted to problems involving regular geometry. The applicability of the method is not confined to regular geometry but it is for this type of problem that the method will have its greatest economic usefulness.

I. INTRODUCTION

The useful life of engineering structures is, on many occasions, highly dependent on the temperature gradients within the structure. A good knowledge of possible temperature distributions is required by the designer of various parts of nuclear reactors, missiles, heat exchangers, and boilers to insure that the thermal stresses resulting from internal temperature differences do not contribute to premature failure.

Frequently the maximum temperature occurring on the surface of a body is of great importance in allowing the designer to choose a material having a suitably high oxidation temperature limit.

Various methods are available to predict temperature distributions in one, two, and three-dimensional structures, each having its peculiar advantages and disadvantages. It then behooves the engineer to use that method which will yield results of sufficient accuracy at the least possible cost.

In a general sense, there are three approaches to determining temperature distributions in solid bodies:

1. An exact method
2. An approximate numerical method
3. An approximate analytical method

It is normally extremely difficult to arrive at exact solutions in heat conduction problems. Exact solutions that

are available for one, two, and three-dimensional problems are complex in form and thus, it is extremely difficult to work with them. As such they are of limited value in the solution of realistic design problems.

The approximate numerical method has proved to be a valuable tool in obtaining temperature distributions in two and three-dimensional bodies. Its principal disadvantage is that in requiring the use of passive element (resistance or resistance-capacitance) analog computers and digital computers, the cost of determining temperatures may become excessive. Also, because of the point by point form of the resulting temperature distributions, carrying out subsequent mathematical operations, such as occur in thermal stress analyses, may become costly since these discrete temperatures may first have to be fit into an equation form.

The great advantage of the approximate analytical method is that, for many problems, it can provide a temperature at any location and at any time without iterative techniques and without manipulation of the surrounding temperature distribution. Such a resultant temperature distribution can easily be mathematically manipulated. A temperature solution which has these particular characteristics is defined as being of "closed form". It is evident that a method providing a "closed form" solution can yield temperature distributions and related thermal stresses much more economically

than can the approximate numerical method or the exact method. Therefore, the approximate analytical method has been studied in detail in this investigation in an attempt to provide suitable ways of applying this concept to find temperature distributions in two and three-dimensional structures.

II. REVIEW OF LITERATURE

Approximate analytical methods have been used successfully in the past few years to find closed-form solutions to one-dimensional heat conduction problems. The three methods which have accomplished this are the integral method, the variational method and Biot's variational method. The published papers which have enhanced the development of these methods or any other analytical techniques which enable one to arrive at approximate temperature distributions in solid bodies, will now be briefly reviewed.

T. von Karman (1) developed the integral-momentum equations in boundary layer theory in 1921. By means of these integral equations, a value for the thickness of the boundary layer was found when Pohlhausen assumed an expression in the form of a polynomial for the velocity profile. The Karman-Pohlhausen approach, (reviewed in Eckert and Drake (2)), because of its use of integral equations, became known as the integral method.

The application of the integral method to the momentum equations consisted of four steps (Figure 1a). (a) Pohlhausen assumed an approximation for the velocity profile of flow over a flat plate. (b) Using boundary conditions, he obtained an expression for the profile in terms of the y distance coordinate and the boundary layer thickness, δ . δ represented the distance from the flat plate surface into

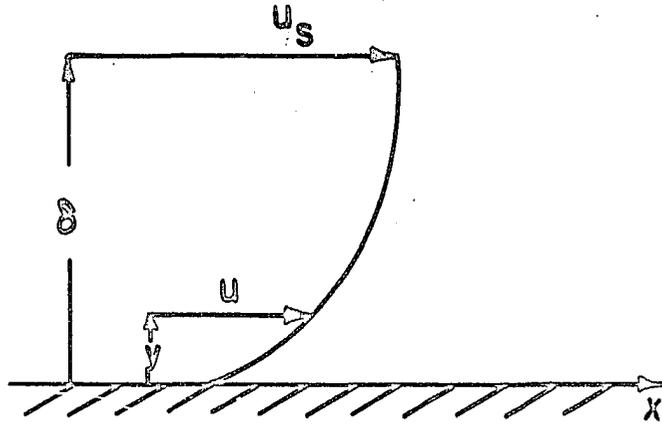


Figure 1a. Laminar boundary layer on a flat plate

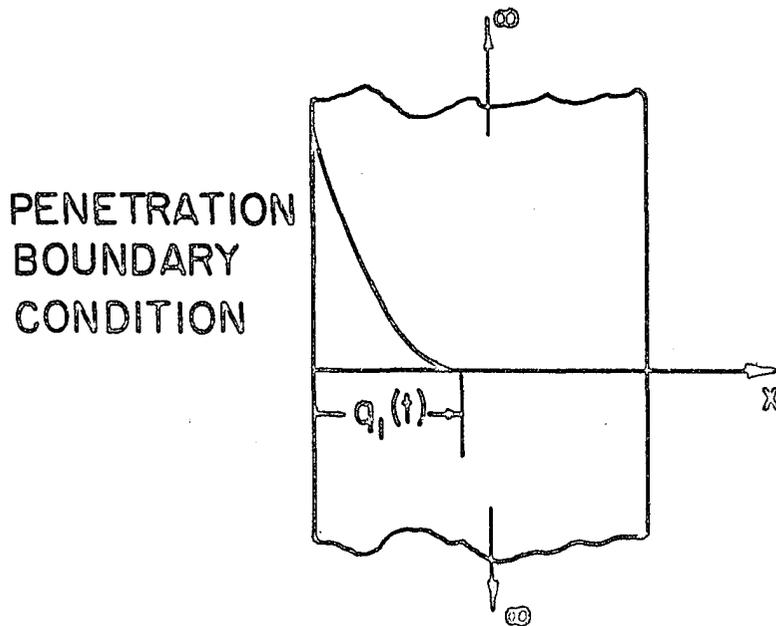


Figure 1b. Temperature penetration depth in a one-dimensional body

the flowing medium, beyond which the velocity gradients were negligible. (c) By substituting the approximate velocity profile into von Karman's integral-momentum equation and integrating it with respect to y , an expression for δ in terms of x (the second distance coordinate) was obtained. (d) Finally, the substitution of δ into the velocity profile resulted in a velocity representation, u , in terms of x and y .

In 1958, Goodman (3) applied the integral method to transient heat conduction in a one-dimensional body. Four steps again describe the procedure (Figure 1b). (a) Goodman assumed a polynomial representation of the temperature distribution inside the body. (b) Using the boundary conditions to restrict and define the distribution, an approximate temperature profile was obtained in terms of the x distance coordinate and the penetration distance, q_1 , which represented the distance into the body beyond which the thermal gradients were negligible. (c) The heat balance integral on the body, with the assumed temperature profile already substituted, was then integrated on x yielding q_1 in terms of the time parameter. (d) Thus, an approximate temperature distribution in terms of distance x and time t was obtained when q_1 was substituted into the temperature profile.

Subsequent to his first publication, Goodman has published a number of applications of the integral method

(4-8). They have been primarily concerned with nonlinear transient heat conduction problems. This means that the integral method provides the ability to consider a problem involving a change of phase, a variation in thermal properties or a radiation boundary condition. In fact, Goodman's first paper (3) required consideration of a change of phase in a semi-infinite solid via the integral method. In a later reference (4), Goodman considers the integral method again applied to a semi-infinite slab and then to a finite slab when the heat flux at the surface of the slab is an arbitrary function of surface temperature and time. For this problem the thermal properties were initially assumed constant and then the method was used in an example when the thermal properties were temperature dependent.

In a 1961 reference (5), Goodman formally extended the concepts of the integral method so that one can include temperature-dependent thermal properties in the analysis. Two final papers by Goodman in conjunction with other authors are concerned with the melting of finite slabs (6) and pulselike heat inputs (7). Goodman's efforts, as well as the efforts of others are reviewed in a section of a book titled "Advances in Heat Transfer" (8). In this book, Goodman (the author of this particular section of the book) not only considers the previous usefulness of the integral method he developed but he also makes some thoughtful

considerations about the theoretical basis of the method, possible ways of improving the accuracy of the method and future applications of this method.

Other papers, many of which are referenced in "Advances in Heat Transfer" (8), that merit attention because of their use of the integral method, will now be mentioned. Reynolds and Dolton (9), were among the first to utilize this technique and they have published an array of simple examples, involving constant thermal properties, that display the advantages of the integral method. A simple transient consideration of a heat exchanger is included among their applications. Yang (10) and Yang and Szewczyk (11) in 1958-59, investigated methods for considering variable thermal properties in semi infinite solids. The later publication utilized the integral method as a basis for this analysis but in a different manner than Goodman has approached the problem. Lardner and Pohle (12) applied the integral technique to one-dimensional problems in cylindrical coordinates. All of the previous analyses had involved one-dimensional problems but this was one of the first considerations outside cartesian coordinates. The authors indicated in this reference that a different type of profile, i.e., different from a polynomial, would improve the accuracy of the method in problems involving nonplanar geometries.

Solutions by Chen (13) and Koh (14) to one-dimensional

problems in heat conduction with arbitrary heating rates may be compared. Koh employed the integral method along with Goodman's transformation for considering thermal properties that are temperature dependent. He also utilized an exponential as an assumed profile to improve on the accuracy of the solution.

Siddall (15) compared finite difference to integral method solutions for a problem in semi-infinite geometry involving constant and variable thermal properties. He concluded that for most practical purposes, the integral method will be adequate and will involve far less labor than any other method of approximate solution.

In 1964, Persson and Persson (16) applied the integral method in a one-dimensional problem where the body was convectively heated. In 1965, Gay (17) compared methods of solution (including the integral method) of the transient one-dimensional heat conduction equation with a radiation boundary condition. So over the years, bodies subjected to the three different types of boundary conditions have been analyzed by integral method techniques. However, all of the publications reviewed have considered one-dimensional heat flow only.

In 1961-62 Yang (18, 19) described an improved integral procedure for considering finite and composite slabs with and without temperature dependent thermal properties. This

was an improvement on the accuracy of the integral method. However, Arpaci (20) has recently developed original concepts on the integral method and on an improvement of the integral method (the variational method). The variational method will normally be more accurate than the integral method for a one-dimensional problem if the same polynomial form is used with both methods. In his book, Arpaci not only considers the concepts of the integral and variational methods but he also applies the techniques in many examples involving one-dimensional heat flow. He further considers examples in two-dimensional steady heat flow by these approximate analytical techniques.

Several other promising approximate analytical methods used to analyze heat conduction problems will now be briefly reviewed. Techniques from the calculus of variations were developed a number of years ago and though utilized in many other fields, were only introduced in heat conduction analysis in the 1950's. Washizu (21), Chambers (22), and Schmit (23) were among the first to apply these methods to conduction heat transfer problems. Later Boswirth (24) used the same techniques on certain simple bodies.

In 1957 Biot (25) applied a different type of variational technique, which he had developed for thermodynamics (26-28), to heat conduction for the first time. His formulation sets up Hamilton's principles of mechanics in an analogous manner

in thermodynamics. Using concepts that are similar to those of integral methods, he (29-31), and others (32-35), utilize the method to solve many practical one-dimensional problems.

All of the heat conduction literature cited so far in this report has been concerned with transient one-dimensional problems. Any of the methods presented would allow one to consider constant or variable thermal properties in a solid and any type of boundary condition on the surface of a body. The methods have all been analytical and the solutions have all been approximate.

Several authors have recognized the need for obtaining approximate temperature distributions in two-dimensional bodies. Weiner (36), Erdogan (37), and Dicker (38) have attempted to solve such problems by taking the Laplace transformation of the time variable to eliminate time and then use Galerkin's¹ technique to find the two-dimensional space variation of the temperature profile. Also within the past few months, Cimprich (39) has applied Biot's variational technique to a two-dimensional problem for the first time. These are the extent of the initial attempts in applying approximate analytical methods in multi-dimensional problems.

¹Galerkin's method is an approximate variational method similar to the Ritz method discussed later in this report.

III. METHODS USED AS BASIS FOR PROPOSED METHOD

Temperature distributions in one-dimensional bodies undergoing transient conduction heat transfer may be found by means of the integral method and the variational method. These are analytical techniques which were mentioned in the "Review of Literature". They are related in terms of a weighting factor (or variation) which has a direct bearing on the accuracy of the methods. The theoretical basis for these methods will be reviewed and then the application of the methods in a typical problem will be outlined.

Penetration
Boundary
Condition

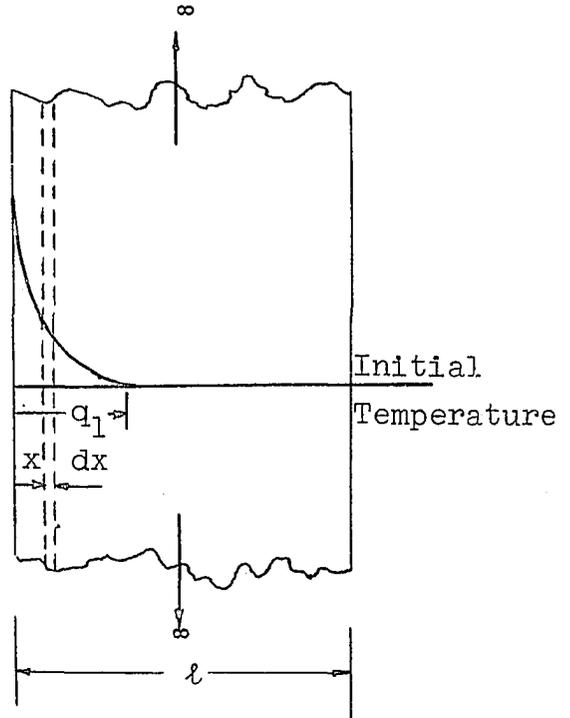


Figure 2. One-dimensional penetration type problem

A. The Integral Method

The integral method will provide approximate temperature distributions, via an analytical analysis, in a body subjected to heat flow of a penetration type, when the flow can

be considered one-dimensional. A penetration type of heat flow problem means that the boundary condition on a particular surface forces heat to flow through that surface and penetrate into the body. If this heat flows in only one direction inside the body, it is considered one-dimensional heat flow.

Let x represent a space dimension, l the thickness of a body and, t the time parameter. Then, it is evident in Figure 2 that heat flows from the $x = 0$ surface toward $x = l$. At a specified time, as a result of the penetration type boundary condition at $x = 0$ heat has penetrated through a certain thickness of the body (beyond this point the temperature in the body has not been significantly changed from its initial temperature). Thus, one can assume that there is a time-dependent "penetration depth" ($q_1(t)$) of heat into the body. Due to this assumption one must now specify time domains for the problem. The time it takes for the heat to penetrate to the $x = l$ surface, i.e., the time needed for this surface to have its temperature changed greatly from the initial body temperature, is known as the penetration time t_{q_1} . All time up to this penetration time is included in the first time domain while the second time domain begins for $t \geq t_{q_1}$.

Any type of penetration boundary condition may be considered by the integral method. Also, the thermal properties of a material can vary with temperature and the

method will still apply. However, throughout this investigation, thermal properties are assumed constant.

Consider the general problem of Figure 2 where T represents temperature and α represents the thermal diffusivity of the material. The first step in applying the integral method to this problem will be either to state the general heat conduction equation explicitly or else derive a form of it from the differential system by means of the first law of thermodynamics (Arpaci (20)). Thus, the integral form of the general heat conduction equation over the depth of the penetration of heat, Equation 1, can be written.

$$\int_{x=0}^{x=q_1} \left(\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T}{\partial t} \right) dx = 0 \text{ for } 0 \leq x \leq q_1, 0 \leq t \leq t_{q_1} \quad (1)$$

Before this integral equation can be integrated, the temperature, T , must be known as a function of distance, x and time, t . So the second step is to assume a mathematical representation of the temperature distribution, for instance, $T(x,t) = \sum_{n=0}^p A_n(t)x^n$. However this distribution must satisfy the boundary conditions of the problem. So the third step is to satisfy the physical conditions of the problem in the assumed profile. It is essential to remember that for the first time domain and at a distance equal to the penetration depth, $q_1(t)$, the temperature is assumed equal to the initial temperature and the thermal gradient at this

point is zero. From these conditions it is noted that one temperature profile may satisfy the conditions of the first time domain but a different profile must be assumed for the second time domain. (A different profile does not necessarily mean that a new mathematical form of the temperature profile must be assumed.) Finally, integrating Equation 1 after substituting the assumed profiles will provide the penetration depth as a function of time, t , for the first time domain and the unknown constant in the temperature profile for the second time domain. An application of this method in a typical problem follows.

The integral method can be used to find the temperature distribution in a plate (Figure 3) considered infinite in two directions and finite with a thickness ℓ in the third dimension, i.e., the x direction. Initially the plate is at zero temperature and suddenly at time, t , the $x = 0$ surface is subjected to a temperature of magnitude T_0 . The $x = \ell$ surface is insulated.

An analysis of the problem may be carried out referring to the differential system of Figure 3 and the first law of thermodynamics. There is no heat generation and no sinks are involved. A simple relation can thus be established between the rate of change of internal energy

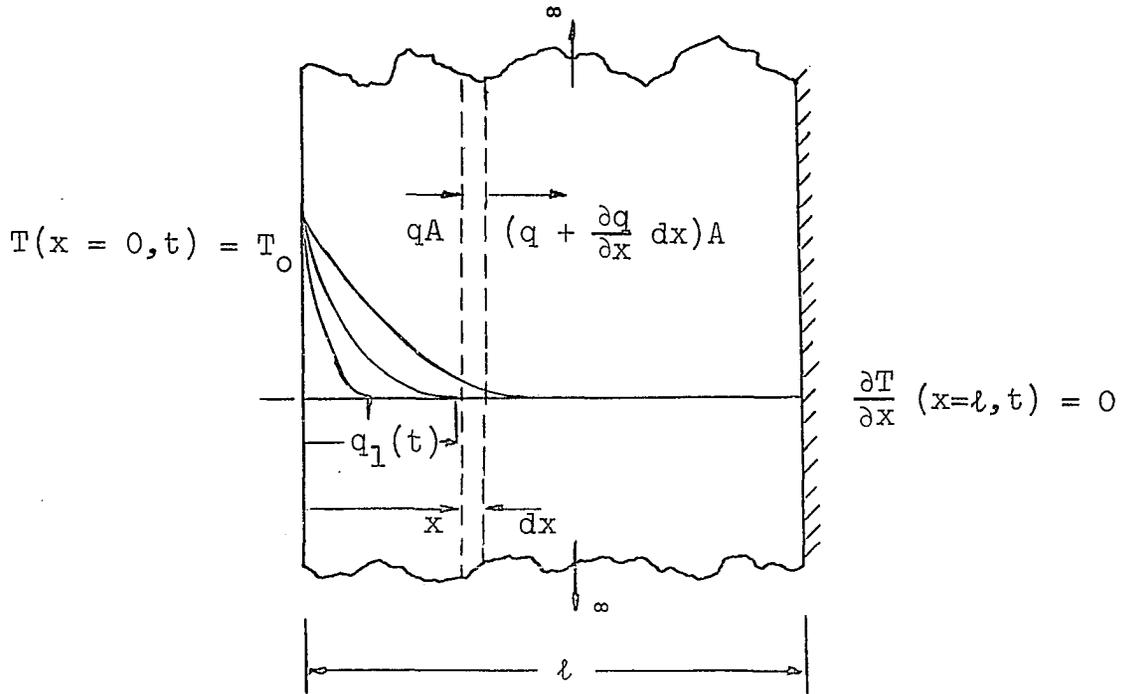


Figure 3. 1-D plate with a step temperature

and the heat conduction through the system. Thus,

$$\frac{d}{dt} [\rho c A T dx] = [-kA \frac{dT}{dx} - \frac{d}{dx} (-kA \frac{dT}{dx}) dx - (-kA \frac{dT}{dx})] \quad (2)$$

The integration of this equation from $x = 0$ to $x = q_1$, with the knowledge that there is no conduction at $x = q_1$, results in Equation 3a.

$$\frac{d}{dt} \int_{x=0}^{x=q_1} \rho c A T dx = kA \left. \frac{dT}{dx} \right|_{x=0} \text{ for } 0 \leq x \leq q_1, 0 \leq t \leq t_{q_1} \quad (3a)$$

Application of the general heat conduction equation (Equation 3b) will yield the same equation after some manipulation, starting however, at a different point.

$$\int_{x=0}^{x=q_1} \left(\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T}{\partial t} \right) dx = 0 \text{ for } 0 \leq x \leq q_1, 0 \leq t \leq t_{q_1} \quad (3b)$$

Before either of these equations can be integrated $T(x,t)$ must be expressed in terms of x . A polynomial with undetermined coefficients is normally assumed for this purpose. With the use of the boundary conditions arrived at via the concept of the penetration depth and the additional regular boundary conditions available, one can establish the coefficients that will define $T(x,t)$. In this particular example, Equation 7 is the proper form for a quadratic temperature profile.

$$T(0,t) = T_0 \text{ for all } t > 0 \quad (4)$$

$$\begin{aligned} T(x = q_1, t) &= 0 \\ \frac{\partial T}{\partial x}(x = q_1, t) &= 0 \end{aligned} \quad \text{for all } 0 \leq t \leq t_{q_1} \quad (5)$$

$$\text{initial condition } q_1(t = 0) = 0 \quad (6)$$

so

$$\begin{aligned} T(x,t) &= \left(T_0 - 2T_0 \frac{x}{q_1} + \frac{T_0 x^2}{q_1^2} \right) = T_0 \left(1 - \frac{x}{q_1} \right)^2 \\ &\text{for } 0 \leq x \leq q_1 \text{ and } t \leq t_{q_1} \end{aligned} \quad (7)$$

Note that this profile will become invalid for $t > t_{q_1}$, i.e., after the first time domain, because the boundary condition (5), will no longer hold. However, in the first time domain by substituting Equation 7 into Equation 3a and solving for the penetration depth, one finds $q_1 = (12\alpha t)^{\frac{1}{2}}$. Thus, for the

first time domain, an approximate temperature profile is given by

$$T(x,t) = T_0 \left(1 - \frac{x}{q_1}\right)^2 \text{ for } 0 \leq t \leq t_{q_1} = \frac{\ell^2}{12\alpha}, \quad 0 \leq x \leq \ell \quad (8)$$

where $q_1 = (12\alpha t)^{\frac{1}{2}}$

In a similar manner, for $t \geq t_{q_1}$, i.e., the second time domain, assuming a quadratic polynomial and using the new boundary and initial conditions

$$\text{B.C. } \frac{\partial T(x = \ell, t)}{\partial x} = 0 \quad \text{for } t > t_{q_1} \quad (9)$$

$$\text{I.C. } T(x, t_{q_1}) = T_0 \left(1 - \frac{x}{\ell}\right)^2 \quad \text{for } 0 \leq x \leq \ell \quad (10)$$

one arrives at Equation 11.

$$T(x,t) = T_0 \left(1 - \frac{x}{\ell}\right)^2 + q_2(t) \left[1 - \left(1 - \frac{x}{\ell}\right)^2\right] \quad (11)$$

Now one must alter Equation 3b so that the integration is carried out over the entire plate.

$$\int_{x=0}^{x=\ell} \left(\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T}{\partial t}\right) dx = 0 \text{ for } 0 \leq x \leq \ell, \quad 0 \leq t_{q_1} \leq t \quad (12)$$

Substitution of Equation 11 into Equation 12 and subsequent integration yields the unknown.

$$q_2(t) = T_0 \left[1 - e^{\left(\frac{1}{4} - \frac{3\alpha t}{\ell^2}\right)}\right] \quad (13)$$

Thus, an approximate temperature distribution for the second time domain has been determined also.

$$T(x,t) = T_0 \left[1 - 2\left(\frac{x}{\ell}\right)e^{-N} + \left(\frac{x}{\ell}\right)^2 e^{-N}\right] \text{ for } 0 \leq x \leq \ell, \quad 0 \leq t_{q_1} \leq t \quad (14)$$

where

$$N = \left[\frac{3\alpha t}{l^2} - \frac{1}{4} \right]$$

A mathematical discussion concerned with the basis of the approximation in the integral method and variational method, which will be discussed shortly, is presented below. The relationship between the two methods is discussed along with the difference between the Ritz and the Kantorovich variational methods.

If one considers the transient heat conduction equation in three dimensions as an operator, B , ($B = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\alpha} \frac{\partial}{\partial t}$) then the T such that $B(T) = 0$, is the exact solution to the equation. When an approximate temperature distribution is operated on, the result becomes $B(T_n) = \epsilon_n$ where the number ϵ_n becomes smaller as the temperature approximation becomes more accurate. In order to help make $\epsilon_n \rightarrow 0$ it has been found to be advantageous to multiply both sides of the equation $B(T_n) = \epsilon_n$ by a weighting factor w_j and average it over all space. Then, if the average $\int w_j \epsilon_n dv$, $j = 1, 2 \dots n$ is set equal to 0, this implies that $\int w_j B(T_n) dv = 0$ for $j = 1, 2, \dots n$. The temperature distribution, T_n , will be taken to satisfy the boundary conditions and to contain n unknown parameters, C_{nj} . When n different weighting factors are chosen, there will be the same number of equations as there are unknowns.

If one chooses a weighting factor $w_j = 1$, then $\int B(T_n)dv = 0$ is seen to be the representation of the integral method. This is not a particularly good choice of a weighting factor. The calculus of variations, in fact, provides a better one.

A weighting factor established by means of the calculus of variations will optimize the general heat conduction integral over the geometry considered. Thus, certain variational techniques can be used as an improvement on the integral method. Approximate variational techniques such as Ritz's or Kantorovich's method may be used for this purpose. In general, these two methods are similar in that they minimize the error between the exact solution and the approximate temperature profile that is obtained in the analysis. In fact, the two methods differ only in the form of the assumed solution. For instance, if the solution is dependent on two variables x_1 and x_2 , a person utilizing the Ritz technique would assume a solution, $\sum_{k=1}^n A_k X_1(x_1) X_2(x_2)$ with $X_1(x_1)$ $X_2(x_2)$ satisfying the boundary conditions. The solution form is thereby set by the form $X_1(x_1)$ $X_2(x_2)$ and only constants, A_k , can be determined. A person utilizing the Kantorovich approach for the same problem would assume a solution $\sum_{k=1}^n f_k(x_1) X_2(x_2)$ with $X_2(x_2)$ satisfying the boundary conditions. In this case, the entire x_1 distribution is unspecified and has to be determined.

B. The Variational Method

For a general one-dimensional penetration heat conduction problem (Figure 2), the procedure for using a variational analysis would require that Equation 1 be only slightly altered. The variational form of that equation would be

$$\int_{x=0}^{x=q_1} \left(\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T}{\partial t} \right) \delta T \, dx = 0 \quad (15)$$

One may observe that the only difference between Equation 1 and Equation 15 is the δT or the variation that is introduced. This so called variation is only useful in that it weighs the integral so that an optimization of the temperature profile is achieved rather than a simple average between the integration limits.

The second step in the application of the variational method will be the same as for the integral method, i.e., a temperature profile has to be assumed. (The particular variational approach chosen will be characterized by the form of this assumed profile as was explained earlier.) With the variational method, as with the integral method, not only are time domains important in the selection of this profile but also the boundary conditions must be satisfied by the profile that is chosen. Substituting the assumed profile into Equation 15 provides the penetration depth as a function of time and the unknown constant in the temperature profile for the first and second time domains respectively. An example of the application of the variational approach in

finding temperature distributions in one-dimensional transient conduction heat transfer problems is presented below.

The variational method will be applied to the same problem (Figure 3) as was considered by the integral method in order to show the distinction between the integral and the variational methods. Since the concepts such as penetration depth and time domain were explained earlier in this thesis, they will not be repeated here.

For the first time domain, a quadratic temperature profile will be assumed as an approximation of the actual temperature distribution.

$$T(x,t) = A(t) + B(t) + C(t)x^2 \quad (16)$$

The boundary conditions and initial conditions for this time domain are simply stated from a knowledge of the penetration depth concept (these are the same as Equations 4 and 5).

Substituting these boundary conditions into the assumed temperature distribution (Equation 16), results in an approximate temperature profile with only one unknown.

$$T(x,t) = \left(T_0 - 2T_0 \frac{x}{q_1} + \frac{T_0 x^2}{q_1^2} \right) = T_0 \left(1 - \frac{x}{q_1} \right)^2$$

for $0 \leq x \leq q_1$ and $t \leq t_{q_1}$ (17)

The particular variational profile approximation used here is the Kantorovich type characterized by the form of the assumed temperature distribution (Equation 17). One of the

variables of the profile, x , is completely specified while the second variable, $q_1(t)$, is left to be determined by the conditions of the problem.

Now this temperature distribution is substituted into the variational form of the integral heat conduction equation (Equation 18) so that a value for the unknown penetration depth $q_1(t)$ can be found.

$$\int_{x=0}^{x=q_1} \left[\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T}{\partial t} \right] \delta T \, dx = 0 \quad (18)$$

Thus, for the first time domain, the temperature profiles arrived at by a variational approach to the problem shown in Figure 3, assuming a quadratic temperature distribution, are given by Equation 19.

$$T(x,t) = T_o \left[1 - \frac{x}{q_1} \right]^2 \quad \text{for } 0 \leq t \leq t_{q_1} = \frac{l^2}{10\alpha}, \quad 0 \leq x \leq l \quad (19)$$

where

$$q_1 = (10\alpha t)^{\frac{1}{2}}.$$

The difference between this temperature distribution and the integral method solution is visible in the value of the penetration depth.

For the second time domain, the initial condition and the boundary conditions have changed.

Boundary Conditions:

1. $T(0,t) = T_o$ for all $t > 0$
2. $\frac{\partial T(x=l,t)}{\partial x} = 0$ for $t > t_{q_1}$ (20)

Initial Condition: $T(x, t_{q_1}) = T_o \left(1 - \frac{x}{l} \right)^2$ for $0 \leq x \leq l$

Once again a temperature profile of the form of Equation 16 is assumed. With conditions (20) restricting such a profile, Equation 16 can then be substituted into the variational Equation 18 and the result will be Equation 21, i.e., the profiles for the second time domain.

$$T(x,t) = T_0 \left\{ 1 - 2\left(\frac{x}{\ell}\right)e^{-N} + \left(\frac{x}{\ell}\right)^2 e^{-N} \right\} \text{ for } 0 \leq x \leq \ell, 0 \leq t_{q_1} \leq t \quad (21)$$

where

$$N = \left[\frac{5\alpha t}{2\ell^2} - \frac{1}{4} \right]$$

Thus, approximate closed form temperature profiles have been found for a typical one-dimensional transient heat conduction problem by the variational method. These profiles can readily be compared to the integral solution to the same problem.

IV. PROPOSED METHOD - THE LAPLACE-VARIATIONAL METHOD

One finds extreme difficulty in attempting to apply the integral method and the variational method successfully to heat conduction problems in more than one dimension. Since their present applicability depends on a penetration depth concept, these methods are beneficial primarily in heat conduction problems which are of the penetration type.

The proposed Laplace-variational method is useful for both penetration and non-penetration type problems, and it lends itself to applications involving multi-dimensional bodies. These points will be brought out in the succeeding discussion along with an outline of the procedure used in applying the method.

The integral method or the variational method theoretically are applicable to two-dimensional problems. The operator B would simply be

$$B = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{\alpha} \frac{\partial}{\partial t} \right) \text{ instead of the one dimensional form,}$$

$$B = \left(\frac{\partial^2}{\partial x^2} - \frac{1}{\alpha} \frac{\partial}{\partial t} \right), \text{ with the integration of B the same in}$$

either case, i.e., $\int w_j B(T_n) dv = 0$. Of course, in applying these methods to two-dimensional problems, temperature profiles, which are dependent on two space co-ordinates and time must be assumed. This is a difficult task. Previously, the penetration depth was simply a function of time, and thus

aided in the description of the temperature profiles, but in two dimensions a surface penetration becomes involved.

$q_1(t)$ which was sufficient to describe the penetration depth in one-dimensional bodies now becomes $q_1 = q_1(x, y, t)$. The two-dimensional problem illustrated in Figure 4, with zero initial temperature, gives an example of the different type penetration depth picture referred to.

Since this is difficult to describe, an approximate method has been developed that will avoid the consideration of such a surface depth while still encompassing the beneficial aspects of a variational consideration. The significant differ-

ence between this new method and the previously presented one-

dimensional methods is the introduction of the Laplace transformation into the analysis. Since the Laplace transformation is used on the time variable, there is no need for the penetration depth analysis. The Laplace transformation on t removes the time variation by transforming

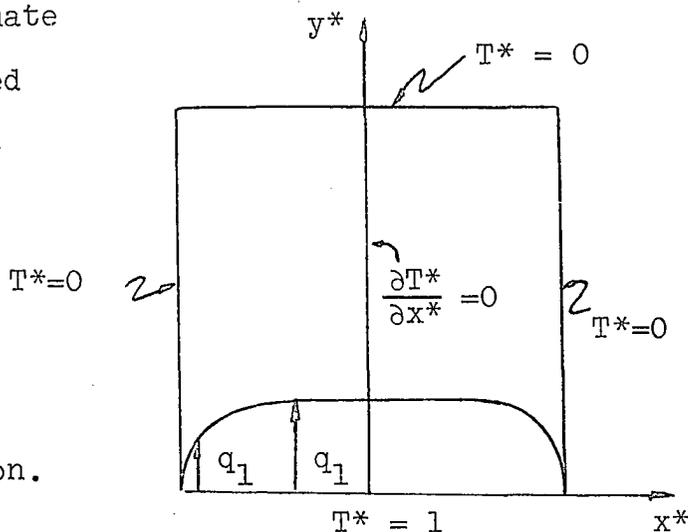


Figure 4. Surface penetration depth for two-dimensional bodies

the problem into the s plane. Thus, the number of independent variables is reduced from three (x , y , and t) to two (x and y). The problem now becomes one of two-dimensional steady state form. So an approximate variational method is utilized in order to arrive at the steady state solution. The Kantorovich variational method was chosen since it results in the greatest accuracy among the approximate methods. The procedure followed in using it is the same as was seen previously in the one-dimensional analysis, only now one of the space variables rather than the time variable is left unrestricted. Thus, one assumes the x direction distribution while the y direction is left to be determined by variational techniques along with the conditions of the problem. After finding this Laplace transformed solution, the Laplace inversion theorem is applied to bring back the time dependence.

The first step required in applying the Laplace-variational method to two-dimensional heat conduction problems is to write the general heat conduction equation for a differential system.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{1}{\alpha} \frac{\partial T}{\partial t} + \frac{q'''}{k} = 0 \quad (22)$$

(Note that with the application of this method heat generation is permissible in the system.) The next step is to apply a Laplace transformation to the time variable. This will alter Equation 22 as well as the boundary conditions of the problem. The variational form of the transformed integral heat

conduction equation can now be written with the limits of integration as the boundaries.

$$\iint \left(\frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} - \frac{s\tilde{T}}{\alpha} + f_1(s) \right) \delta\tilde{T} dx dy = 0 \quad (23)$$

The temperature as a function of the space co-ordinates, x and y , must be known if the integration of Equation 23 is to be carried out. The assumption of an approximate temperature distribution of the Kantorovich variational form will serve this purpose. To do this, the spatial distribution in one co-ordinate direction is assumed, consistent with the boundary conditions, while the other co-ordinate direction is left unrestricted. The substitution of this distribution into Equation 23 and the subsequent integration with respect to the restricted direction will yield a second-order differential equation. This equation will then be solved, and by requiring the solution to satisfy the boundary conditions, the result will be a Laplace transformed temperature distribution. Use of the Inversion Theorem for Laplace transformations on this transformed distribution provides a closed form approximate temperature distribution to the two-dimensional problem.

The greatest difficulty inherent in the utilization of the Laplace-variational method is connected with the choice of an assumed temperature distribution which is required for beginning the solution to a problem. Normally there will be several different forms of assumed Kantorovich type temperature

distributions which will satisfy the boundary conditions of a problem. The best profile will be chosen from among these on the basis of the experience of the investigator. For instance, it has been found that for rectangular geometry, circular profiles (trigonometric functions) provide the best solution. However, when a constant heat generation affects an entire rectangular plate, a polynomial representation will provide the most accurate temperature distribution.

Subsequent to Goodman's initial publication on the use of the integral method in heat conduction, many investigators have been able to improve on the type of assumed temperature profile which will give the best accuracy for a particular heat conduction problem. In the same way, persons who utilize the Laplace-variational method will be able to recommend the use of certain types of assumed temperature profiles for analyzing specific heat conduction problems on the basis of their experience with the method.

V. APPLICATIONS OF THE LAPLACE-VARIATIONAL METHOD

The Laplace-variational method can be used to solve for temperature distributions in many types of heat conduction problems. Two different types of problems will now be analyzed in order to show the range of applicability of this method.

A. The Laplace-Variational Solution of a Penetration Type Problem

A penetration type problem (Figure 5) will be solved by the Laplace-variational method. The example chosen involves a two-dimensional rectangular plate which is initially at zero temperature at every point including the boundaries.

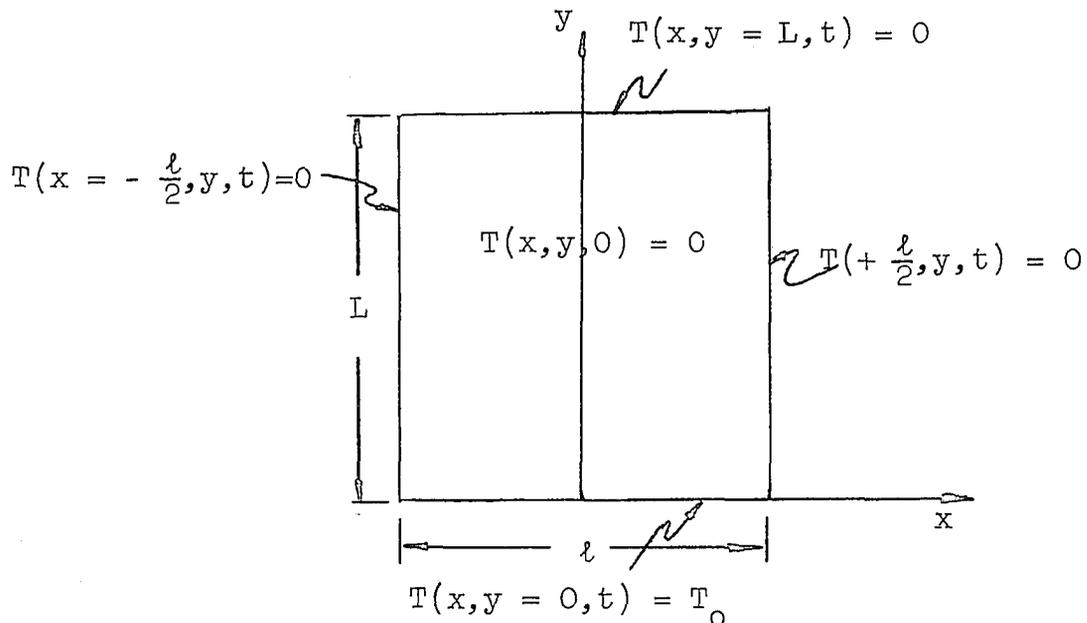


Figure 5. 2-D plate with a step temperature

At time t , the $y = 0$ surface is subjected to a step change in temperature, T_0 .

The conditions on the problem can be stated mathematically as follows.

$$\text{Initial Condition: } T(x,y,0) = 0$$

Boundary Conditions:

$$\begin{aligned} \text{(a) } T(x,y = L,t) &= 0 & \text{(c) } \frac{\partial T}{\partial x} (x=0,y,t) &= 0 \\ \text{(b) } T(x = \pm \frac{l}{2}, y,t) &= 0 & \text{(d) } T(x,y = 0,t) &= T_0 \end{aligned} \quad (24)$$

The general heat conduction equation for two-dimensional heat flow will be used in this analysis.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{1}{\alpha} \frac{\partial T}{\partial t} = 0 \quad (25)$$

The Laplace transformation on the time variable of $T(x,y,t)$ is defined by Equation 26 (Carslaw and Jaeger (40)). Other references that may aid in applying this transformation are Churchill (41), Doetsch (42) and Boley and Weiner (43).

$$\tilde{T}(x,y,s) = \int_0^{\infty} T(x,y,t)e^{-st} dt \quad (26)$$

The Laplace transformed governing equation and boundary conditions can now be written.

$$\frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} - \frac{s\tilde{T}}{\alpha} = 0 \quad (27)$$

Boundary Conditions:

$$\begin{aligned} \text{(a) } \tilde{T}(x,y = L,s) &= 0 & \text{(c) } \frac{\partial \tilde{T}}{\partial x} (x=0,y,s) &= 0 \\ \text{(b) } \tilde{T}(x = \pm \frac{l}{2}, y,s) &= 0 & \text{(d) } \tilde{T}(x,y=0,s) &= T_0/s \end{aligned} \quad (28)$$

Following the procedure of the Laplace-variational (Kantorovich) method a temperature distribution that satisfies the boundary conditions will now be assumed.

$$\tilde{T}(x,y,s) = \left(\cos \frac{\pi x}{l}\right) Y(y,s) \quad (29)$$

The x direction distribution was assumed to be $\left(\cos \frac{\pi x}{l}\right)$ on the basis of the type of boundary conditions involved. At $x = 0$, the boundary condition indicates that the derivative of the function with respect to x is 0, i.e., $\frac{d}{dx} \left(\cos \frac{\pi x}{l}\right) = \frac{\pi}{l} \left(\sin \frac{\pi x}{l}\right) = \left(\sin 0\right) = 0$. At $x = l/2$, the function itself must be 0, i.e., $\left[\cos \frac{\pi}{l} \left(\frac{l}{2}\right)\right] = \left(\cos \frac{\pi}{2}\right) = 0$. Since both of these are well known characteristics of the cosine trigonometric function, this seems to be an obvious choice for the x distribution. However, other reasonable choices would be a polynomial or an exponential distribution. Either $\left[\left(\frac{l}{2}\right)^2 - x^2\right]$ or $\left[1 - e^{-\left(x^2 - \left(\frac{l}{2}\right)^2\right)}\right]$ will satisfy the x direction boundary conditions. One must then rely on experience to dictate a choice. As was stated earlier in this thesis, after sufficient experience, it has been found that circular functions (trigonometric functions) are normally more accurate for the solution of problems of rectangular geometry when heat generation is not involved. Thus, this is the rule followed here in the choice of a temperature distribution (Equation 29).

In order to find the best temperature distribution of this form, i.e., the best representation for $Y(y,s)$, one

must first write an integral consistent with the physics of the problem (Equation 30).

$$2 \int_{x=0}^{\frac{\ell}{2}} \int_{y=0}^{y=L} \left\{ \frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} - \frac{s\tilde{T}}{\alpha} \right\} dy dx = 0 \quad (30)$$

Then this equation can be written in a simple Euler equation variational form (Arpaci (20, Chap. 8)).

$$2 \int_{y=0}^{y=L} \int_{x=0}^{\frac{\ell}{2}} \left(\frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} - \frac{s\tilde{T}}{\alpha} \right) \delta \tilde{T} dx dy = 0 \quad (31)$$

It can readily be observed that to be able to integrate Equation 31, one must know the temperature as a function of x and y . Equation 29 provides this assumed temperature distribution. Operating on this assumed profile one can arrive at the quantities required for integration of Equation 31.

$$\frac{\partial^2 \tilde{T}}{\partial x^2} = - \frac{\pi^2}{\ell^2} \left(\cos \frac{\pi x}{\ell} \right) Y$$

$$\frac{\partial^2 \tilde{T}}{\partial y^2} = \left(\cos \frac{\pi x}{\ell} \right) Y''$$

By definition of the variation:

$$\delta \tilde{T} = \frac{\partial \tilde{T}}{\partial y} \delta Y = \left(\cos \frac{\pi x}{\ell} \right) \delta Y$$

Substitution of these quantities into Equation 31 will give Equation 32.

$$2 \int_{y=0}^{y=L} \int_{x=0}^{\frac{\ell}{2}} \left\{ \left(\cos \frac{\pi x}{\ell} \right) Y'' - \left(\frac{\pi^2}{\ell^2} + \frac{s}{\alpha} \right) \left(\cos \frac{\pi x}{\ell} \right) Y \right\} \left(\cos \frac{\pi x}{\ell} \right) \delta Y dx dy = 0 \quad (32)$$

This equation can now be integrated

$$\int_{y=0}^{y=L} \left\{ - \left(\frac{\pi^2}{l^2} + \frac{s}{\alpha} \right) Y + Y'' \right\} \delta Y dy = 0 \quad (33)$$

Since δY is arbitrary (Elsgolc (44), Arpaci (20), Sokolnikoff (45) and Kantorovich (46)) the integrand can be set equal to 0 and solved as a simple second order differential equation.

$$Y(y, s) = A e^{my} + B e^{-my} \quad (34)$$

where

$$m = \left(\frac{\pi^2}{l^2} + \frac{s}{\alpha} \right)^{\frac{1}{2}}$$

Substitution of the boundary conditions at $y = 0$ and $y = L$ allows one to find $Y(y, s)$ explicitly.

$$Y(y, s) = \frac{T_0}{s} \frac{\sinh m(L-y)}{\sinh mL} \quad (35)$$

Thus, the assumed temperature profile is now given by Equation 36.

$$\tilde{T}(x, y, s) = \left(\cos \frac{\pi x}{l} \right) \left(\frac{T_0}{s} \right) \frac{\sinh \left[\left(\frac{\pi^2}{l^2} + \frac{s}{\alpha} \right)^{\frac{1}{2}} (L-y) \right]}{\sinh \left[\left(\frac{\pi^2}{l^2} + \frac{s}{\alpha} \right)^{\frac{1}{2}} L \right]} \quad (36)$$

The inverse transformation of the approximate temperature function back to the real time plane is defined (Carslaw and Jaeger (40) and Churchill (41)) in Equation 37.

$$T(x, y, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{T}(x, y, s) e^{st} ds \quad (37)$$

When the temperature distribution (Equation 36) is substituted into this complex integral it can be seen that the integral has poles for zeros of:

(1) $s=0$ (This simple pole at the origin in the s plane will lead to a steady-state temperature term in the real time plane. (Arpaci (20))

$$(2) \quad \sinh \left[\frac{\pi^2}{\ell^2} + \frac{s}{\alpha} \right]^{\frac{1}{2}} L = 0 \quad \rightarrow \quad \frac{e^{mL} - e^{-mL}}{2} = 0$$

$$\text{where} \quad m = \left[\frac{\pi^2}{\ell^2} + \frac{s}{\alpha} \right]^{\frac{1}{2}} \rightarrow$$

$$e^{2mL} = 1 \rightarrow e^{2mL} = e^{i(2n\pi)} \quad \text{for } n = 1, 2, \dots$$

for $n = 0$, there is no pole involved.

$$2mL = 2n\pi i \rightarrow m^2 L^2 = -n^2 \pi^2$$

$$\left(\frac{\pi^2}{\ell^2} + \frac{s}{\alpha} \right) = -\frac{n^2 \pi^2}{L^2} \rightarrow s = -\alpha \left[\frac{n^2 \pi^2}{L^2} + \frac{\pi^2}{\ell^2} \right]$$

(These simple poles will lead to exponentially decreasing temperature terms in the real time plane (Arpaci (20).)

No branch cut is needed for this problem. The inversion can be carried out simply by using the Residue Theorem at each of the poles.

$$T(x, y, t) = \text{Residue}(s=0) + \sum_{n=1}^{\infty} \text{Residue} \left[-\alpha \left(\frac{n^2 \pi^2}{L^2} + \frac{\pi^2}{\ell^2} \right) \right] \quad (38)$$

So

$$T(x, y, t) = T_0 \left(\cos \frac{\pi x}{\ell} \right) \frac{[\sinh \frac{\pi(L-y)}{\ell}]}{[\sinh \frac{\pi L}{\ell}]}$$

$$+T_0 \left(\cos \frac{\pi x}{l} \right) \frac{2}{\pi} \sum_{n=1}^{\infty} n(-1)^n \left[\sin \left(\frac{n\pi}{L} \right) (L-y) \right] \frac{e^{-\alpha \left[\frac{n^2}{L^2} + \frac{1}{l^2} \right] \pi^2 t}}{L^2 \left(\frac{n^2}{L^2} + \frac{1}{l^2} \right)} \quad (39)$$

Five dimensionless parameters may be defined for this analysis.

$$T^* = \frac{T(x,y,t)}{T_0}, \quad x^* = \frac{x}{l}, \quad y^* = \frac{y}{l}, \quad t^* = \frac{\alpha t}{l^2}, \quad p = \frac{L}{l} \quad (40)$$

Thus, the dimensionless temperature distribution is given by Equation 41.

$$T^* (x^*, y^*, t^*) = (\cos \pi x^*) \left[\frac{\sinh \pi (p-y^*)}{(\sinh \pi p)} \right] + \frac{2}{\pi} (\cos \pi x^*) \sum_{n=1}^{\infty} (-1)^n n \left[\sin(n\pi) \left(1 - \frac{y^*}{p} \right) \right] \frac{e^{-\pi^2 \left[1 + \frac{n^2}{p^2} \right] t^*}}{(n^2 + p^2)} \quad (41)$$

Thus, an approximate temperature distribution to a two-dimensional transient heat conduction problem, involving a step change at one surface and homogeneous temperatures at the other surfaces of a rectangular plate has been found. The exact solution to this same problem involves a double infinite series and is given by Equation 42.

$$T(x,y,t) = \left(\frac{4T_0}{\pi} \right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \left[\cos \left(\frac{(2n+1)\pi x}{l} \right) \right] \frac{\sinh \left[\frac{(2n+1)\pi}{L} (L-y) \right]}{\sinh \frac{(2n+1)\pi L}{l}}$$

$$\begin{aligned}
& + \frac{8T_0}{\pi^2} \sum_{n=0}^{\infty} \sum_{n_1=1}^{\infty} \\
& \frac{(-1)^{n+n_1} n_1}{L^2(2n+1)} \frac{[\cos(2n+1)\frac{\pi x}{L}][\sin\frac{n_1\pi(L-y)}{L}]e^{-\alpha(\frac{n_1^2\pi^2}{L^2} + \frac{(2n+1)^2\pi^2}{L^2})t}}{\{\frac{n_1^2}{L^2} + \frac{(2n+1)^2}{L^2}\}}
\end{aligned} \tag{42}$$

By using the defined dimensionless parameters (40) the exact temperature distribution can be written in dimensionless form.

$$\begin{aligned}
T^*(x^*, y^*, t^*) &= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} [\cos(2n+1)\pi x^*] \left\{ \frac{\sinh(2n+1)\pi(p-y^*)}{\sinh(2n+1)\pi p} \right\} \\
& + \frac{8}{\pi^2} \sum_{n=0}^{\infty} \sum_{n_1=1}^{\infty} \\
& \frac{(-1)^{n+n_1} n_1 [\cos(2n+1)\pi x^*] [\sin n_1\pi(1 - \frac{y^*}{p})] e^{-\pi^2(\frac{n_1^2}{p^2} + (2n+1)^2)t^*}}{(2n+1) [n_1^2 + (2n+1)^2 p^2]}
\end{aligned} \tag{43}$$

A comparison of this exact solution to the approximate one-term Laplace-variational solution (Equation 41) is made in the "Discussion of the Results".

On many occasions, a Laplace-variational profile may be chosen so that the restricted portion of the assumed Kantorovich temperature profile is a more accurate representation of the solution than some other assumption might be. The Laplace-variational method will then give a more accurate

resultant temperature distribution. It can be shown that the final temperature distribution will be more accurate, using the Laplace-variational method, if an improvement is made on the initial assumed temperature distribution to a problem. The procedure for doing so follows.

The more exactly the x direction-specified temperature distribution is represented, the better the overall temperature distribution will be. Therefore, rather than assuming that the direction distribution is simply $(\cos \frac{\pi x}{l})$, one can assume an infinite number of eigenfunctions that will satisfy the homogeneous boundary conditions of this direction. The x direction boundary conditions are obviously homogeneous and thus will be satisfied by either the trivial $x = 0$ solution or by eigenfunctions. The following eigenfunctions $\sum_{n=0}^{\infty} A_n (\cos (2n+1) \frac{\pi x}{l})$ satisfy these boundary conditions for all n. Requiring that this series be orthogonal and convergent while leaving the Y direction unspecified will lead to the choice of Equation 44 to be the assumed Kantorovich temperature profile.

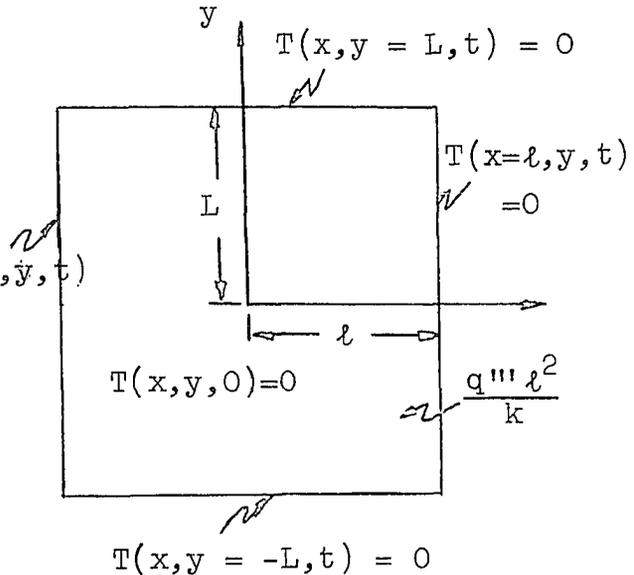
$$\tilde{T}(x,y,s) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)\pi} [\cos \frac{(2n+1)\pi x}{l}] Y(y,s) \quad (44)$$

The same procedure of operation is carried out on Equation 44 as was followed for the first approximation (Equation 29) and the result turns out to be the exact solution (Equation 43). So proper accuracy can be attained via this new

combination Laplace-variational approximate method.

B. The Laplace-Variational Solution of a Non-Penetration (Heat Generation) Type Problem

The Laplace-variational method can be used to determine temperature distributions in a rectangular plate with heat generation (Figure 6) throughout and homogeneous temperature boundaries (i.e., $T(x,y,t) = 0$ at the surface of the plate).



The analysis will be carried out in dimensional units and will be put in dimensionless form

Figure 6. 2-D plate with heat generation

after the temperature distributions have been developed.

The nonhomogeneous governing equation is:

$$\frac{\partial^2 T}{\partial x^2}(x,y,t) + \frac{\partial^2 T}{\partial y^2}(x,y,t) - \frac{1}{\alpha} \frac{\partial T(x,y,t)}{\partial t} + \frac{q''''}{k} = 0 \quad (45)$$

Initial Condition: $T(x,y,0) = 0$

Boundary Conditions:

$$\begin{aligned} (a) \quad T(x,y = L,t) &= 0 & (c) \quad \frac{\partial T(x = 0,y,t)}{\partial x} &= 0 \\ (b) \quad T(x = l,y,t) &= 0 & (d) \quad \frac{\partial T(x,y = 0,t)}{\partial y} &= 0 \end{aligned} \quad (46)$$

Define the Laplace transformation (Doetsch (42)).

$$\tilde{T}(x,y,s) = \int_0^{\infty} T(x,y,t)e^{-st} dt \quad (47)$$

Thus, the Laplace transformed governing equation and boundary conditions are:

$$\frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} - \frac{s\tilde{T}}{\alpha} + \frac{q'''}{sk} = 0 \quad (48)$$

Boundary Conditions

$$\begin{aligned} (a) \quad \tilde{T}(x,y=L,s) &= 0 & (c) \quad \frac{\partial \tilde{T}(x=0,y,s)}{\partial x} \\ (b) \quad \tilde{T}(x=l,y,s) &= 0 & (d) \quad \frac{\partial \tilde{T}(x,y=0,s)}{\partial y} \end{aligned} \quad (49)$$

Assume a temperature distribution in the form of a Kantorovich profile.

$$\tilde{T}(x,y,s) = (L^2 - y^2) \cdot X(x,s) \quad (50)$$

Note that this profile satisfies the boundary condition at $y = L$ as well as the symmetry of the problem (only even terms are used). It has been found by experience that this is the best form to be chosen for a temperature profile when a rectangular plate has heat generation.

In order to find the best temperature distribution of this form, one must first write an integral consistent with the physics of the problem (Equation 51).

$$\int_{y=-L}^{y=L} \int_{x=-l}^{x=l} \left\{ \frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} - \frac{s\tilde{T}}{\alpha} + \frac{q'''}{sk} \right\} dx dy = 0 \quad (51)$$

With reference to Arpaci ((20), Chap. 8), Equation 51 can be rewritten in simple Euler equation variational form.

$$4 \int_{y=0}^{y=L} \int_{x=0}^{x=l} \left[\frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} - \frac{s\tilde{T}}{\alpha} + \frac{q'''}{sk} \right] \delta \tilde{T} dx dy = 0 \quad (52)$$

The terms in Equation 52 are evaluated in terms of the Kantorovich profile (Equation 50):

$$\frac{\partial^2 \tilde{T}}{\partial x^2} = (L^2 - y^2) X'' \quad \text{and} \quad \frac{\partial^2 \tilde{T}}{\partial y^2} = -2X$$

From the definition of the variation:

$$\delta \tilde{T} = \frac{\partial \tilde{T}}{\partial X} \cdot \delta X = (L^2 - y^2) \delta X$$

If these terms are substituted into Equation 52:

$$4 \int_{y=0}^{y=L} \int_{x=0}^{x=l} \left[(L^2 - y^2) X'' - 2X - \frac{s(L^2 - y^2)}{\alpha} X + \frac{q'''}{sk} \right] (L^2 - y^2) \delta X dx dy = 0 \quad (53)$$

The integration of this equation results in:

$$\int_{x=0}^{x=l} \left[X'' - \left(\frac{5}{2L^2} + \frac{s}{\alpha} \right) X + \left(\frac{5q'''}{4L^2 k} \right) \frac{1}{s} \right] \delta X dx = 0 \quad (54)$$

Since δX is arbitrary, the integrand in Equation 54 is set = 0. Thus,

$$X'' - \left(\frac{5}{2L^2} + \frac{s}{\alpha} \right) X + \left(\frac{5q'''}{4L^2 k} \right) \frac{1}{s} = 0 \quad (55)$$

The homogeneous and particular solutions to this differential equation are found and when combined, they provide a general solution on x .

$$X(x) = Ae^{-\left(\frac{5\alpha+2sL^2}{2\alpha L^2}\right)^{\frac{1}{2}} x} + Be^{\left(\frac{5\alpha+2sL^2}{2\alpha L^2}\right)^{\frac{1}{2}} x} + \frac{\left(\frac{5q'''}{4L^2 k}\right) \left(\frac{1}{s}\right)}{\left(\frac{5\alpha+2sL^2}{2\alpha L^2}\right)} \quad (56)$$

The boundary conditions in the x direction are:

$$\frac{dX}{dx} (x = 0) = 0 \quad \text{and} \quad X(x = l) = 0 \quad (57)$$

These conditions permit one to find constants A and B of Equation 56. Substituting the values for A and B into Equation 56 yields the following equation:

$$X(x) = \frac{5q''' \alpha}{2k} \left(\frac{1}{s(5\alpha + 2sL^2)} \right) \frac{\cosh \left(\frac{5\alpha + 2sL^2}{2\alpha L^2} \right)^{\frac{1}{2}} x}{\cosh \left(\frac{5\alpha + 2sL^2}{2\alpha L^2} \right)^{\frac{1}{2}} l} - \frac{5q''' \alpha}{2k} \left(\frac{1}{s(5\alpha + 2sL^2)} \right) \quad (58)$$

Thus, the Laplace transformed approximate temperature is:

$$\tilde{T}(x, y, s) = (L^2 - y^2) \left[\frac{5q''' \alpha}{2k} \left(\frac{1}{s(5\alpha + 2sL^2)} \right) \frac{\cosh \left(\frac{5\alpha + 2sL^2}{2\alpha L^2} \right)^{\frac{1}{2}} x}{\cosh \left(\frac{5\alpha + 2sL^2}{2\alpha L^2} \right)^{\frac{1}{2}} l} - \frac{5q''' \alpha}{2k} \left(\frac{1}{s(5\alpha + 2sL^2)} \right) \right] \quad (59)$$

The inverse transformation (Carslaw and Jaeger (40), p. 322) of this approximate temperature is given by:

$$T(x, y, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{T}(x, y, s) e^{st} ds \quad (60)$$

This complex integral has poles for zeros of:

- (1) $s = 0$ (This simple pole at the origin will lead to a steady-state temperature term. (Arpaci (20), Chap. 7)).

(2) $(5\alpha + 2sL^2) = 0 \Rightarrow s = -\frac{5\alpha}{2L^2}$ (This simple pole on negative real axis will lead to an exponentially decreasing temperature term.)

$$(3) \cosh \left(\frac{5\alpha + 2sL^2}{2\alpha L^2} \right)^{\frac{1}{2}} l = 0 \Rightarrow \left[\frac{e^{ml} + e^{-ml}}{2} \right] = 0$$

$$\text{where } m = \left(\frac{5\alpha + 2sL^2}{2\alpha L^2} \right)^{\frac{1}{2}}$$

$$\text{So } e^{2ml} = -1 \Rightarrow e^{2ml} = e^{(2n+1)\pi i} \text{ for } n = 0, 1, \dots$$

$$2ml = (2n+1)\pi i$$

Substituting for m and squaring both sides of this equation will result in

$$\left[\frac{10l^2}{L^2} + \frac{4l^2 s}{\alpha} \right] = -\pi^2 (2n+1)^2$$

$$\text{So } s = \left(-\frac{\alpha}{4l^2} \right) \left[-\frac{10l^2}{L^2} - \pi^2 (2n+1)^2 \right] \text{ for } n = 0, 1, \dots$$

(These simple poles all the way out the negative real axis will lead to an exponentially decreasing temperature term for each n.)

Now the Residue Theorem (Churchill (41), p. 159 and 186) will be used to enable one to find the approximate transient temperature distribution. No branch cut needs to be considered for this analysis. Thus, knowing this theorem one can write:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{T}(x, y, s) e^{st} ds = \text{Residue}[s=0] + \text{Residue} \left[s = -\frac{5\alpha}{2L^2} \right]$$

$$+ \sum_{n=0}^{\infty} \text{Residue} \left[s = -\left(\frac{\alpha}{4\ell^2}\right) \left[\frac{10\ell^2}{L^2} + \pi^2(2n+1)^2 \right] \right] \quad (61)$$

The residues at the poles can be evaluated. The results can be simplified if one knows that:

$$\begin{aligned} \cosh ix &= \cos x \\ \sinh ix &= i \sin x \end{aligned} \quad (62)$$

one finds:

$$\begin{aligned} T(x,y,t) &= \frac{q'''(L^2-y^2)}{2k} \left[\frac{\cosh \left[\left(\frac{5}{2}\right)^{\frac{1}{2}} \frac{x}{L} \right]}{\cosh \left[\left(\frac{5}{2}\right)^{\frac{1}{2}} \frac{\ell}{L} \right]} \right. \\ &- 40 \sum_{n=0}^{\infty} \frac{(-1)^n \cos \left[\frac{(2n+1)\pi}{2} \left(\frac{x}{\ell}\right) \right]}{[(2n+1)[10+(2n+1)^2\pi^2\left(\frac{\ell}{L}\right)^2]} e^{\left[-\frac{5}{2} + \frac{\pi^2(2n+1)^2L^2}{4\ell^2} \right] \frac{\alpha t}{L^2}} \left. \right] \quad (63) \end{aligned}$$

Because of symmetry this can be rewritten as:

$$\begin{aligned} T(x,y,t) &= \frac{q'''(\ell^2-x^2)}{2k} \left[\frac{\cosh \left[\left(\frac{5}{2}\right)^{\frac{1}{2}} \frac{y}{\ell} \right]}{\cosh \left[\left(\frac{5}{2}\right)^{\frac{1}{2}} \frac{L}{\ell} \right]} \right. \\ &- 40 \sum_{n=0}^{\infty} \frac{(-1)^n \cos \left[\frac{(2n+1)\pi}{2} \left(\frac{y}{L}\right) \right]}{[(2n+1)[10+(2n+1)^2\pi^2\left(\frac{\ell}{L}\right)^2]} e^{\left[-\frac{5}{2} + \frac{\pi^2(2n+1)^2\ell^2}{4L^2} \right] \frac{\alpha t}{\ell^2}} \left. \right] \quad (64) \end{aligned}$$

Let:

$$T^*(x^*,y^*,t^*) = \frac{T(x,y,t)}{\frac{q''' \ell^2}{k}} \quad (65)$$

$$x^* = \frac{x}{\ell}, \quad y^* = \frac{y}{L}, \quad \text{and} \quad t^* = \frac{\alpha t}{\ell^2}$$

Substituting the dimensionless expressions from Equation 65 into Equation 64, the dimensionless temperature distribution can be written as follows:

$$T^*(x^*, y^*, t^*) = \frac{1}{2} (1-x^{*2}) \left[1 - \frac{\cosh \left[\left(\frac{5}{2}\right)^{\frac{1}{2}} y^* \right]}{\cosh \left[\left(\frac{5}{2}\right)^{\frac{1}{2}} \left(\frac{L}{l}\right) \right]} - \left(\frac{5}{2} + \frac{\pi^2 (2n+1)^2 l^2}{4L^2}\right) t^* \right] e^{-\frac{(-1)^n \cos \left[\frac{(2n+1)\pi}{2} \left(\frac{l}{L}\right) y^* \right]}{[(2n+1)\pi \{10 + (2n+1)^2 \pi^2 \left(\frac{l}{L}\right)^2\}]} t^*} \quad (66)$$

Six terms of this series will normally provide a sufficient representation of this approximate distribution.

For comparison, the exact dimensionless solution of the same problem was found using separation of variables along with superposition.

$$T^*(x^*, y^*, t^*) = \frac{1}{2} (1-x^{*2}) - \frac{16}{\pi^3} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1} \left[\cos \left(\frac{2n_1+1}{2} \pi x^* \right) \right] \left[\cosh \left(\frac{2n_1+1}{2} \pi y^* \right) \right]}{(2n_1+1)^3 \left[\cosh \left(\frac{2n_1+1}{2} \right) \frac{\pi L}{l} \right]} - \frac{64}{\pi^4} \sum_{n_1=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{n_1+n} \left[\cos \frac{2n_1+1}{2} \pi x^* \right] \left[\cos \frac{2n+1}{2} \frac{\pi l}{L} y^* \right] e^{-t^* \left[\frac{(2n+1)^2 \pi^2 l^2}{4L^2} + \frac{(2n_1+1)^2 \pi^2}{4} \right]}}{(2n_1+1)(2n+1) \left[\frac{(2n+1)^2 l^2}{L^2} + (2n_1+1)^2 \right]} \quad (67)$$

VI. DISCUSSION OF THE RESULTS

Integral and variational methods provide approximate temperature distributions with good accuracy compared to exact solutions (Carslaw and Jaeger (40), Churchill (41), Doetsch (42), and Schneider (47)) for one-dimensional transient conduction heat transfer problems. This can be noted in references (3-20) and also on the basis of the one typical example worked previously in this thesis. Figures 8 and 9 substantiate that claim. It can also be observed from these two figures that the variational method temperature distribution is normally more accurate than the integral method solution to a typical problem. However, even though all of these facts are true for one-dimensional heat conduction, neither of these methods has ever been utilized with success in analyzing a problem involving two-dimensional heat flow. The reason for this is obvious if the reader is familiar with the two methods. To assume temperature profiles in two space coordinates on a time basis is very difficult and that is precisely what the use of these methods would require. Even a penetration depth picture will not help since the simple dependence of penetration on time, as was true in one-dimensional problems, now becomes a surface depth picture, dependent on two space variables and time.

Therefore, a Laplace-variational method was developed which will provide approximate closed form temperature

distributions for two-dimensional transient conduction heat transfer problems. This method avoids the need for a surface penetration depth picture by utilizing a Laplace transformation on the time variable, thus effectively eliminating time from the analysis. It simplifies the analysis of the spatial temperature distribution by allowing one space coordinate to be completely unrestricted.¹ However, by leaving this one coordinate free, good accuracy is required in assuming the second spatial coordinate distribution if the final accuracy of the approximate temperature profiles is to be favorable. This is clearly seen in Figures 10-12, which show some temperature distributions determined for the two-dimensional problem considered in Section VA. Temperature contour maps (Figures 13-14) give a physical picture of how the heat flows in the problem. For this simple penetration type example a 1-term Laplace-variational approximation is not very accurate but when two terms are considered the accuracy improves noticeably. Of course, in this particular problem, there is a discontinuity at $(x^* = \pm 0.5, y^* = 0)$ and thus the greatest inaccuracy is noticeable near that specific point (Figure 10).

If greater accuracy is required with the Laplace-variational method, the temperature profile must be assumed

¹This is done by using a Kantorovich variational analysis on the problem. (Hence the name Laplace-variational method.)

more exactly in the restricted direction of the spatial distribution. It is shown in an application of the method (Section VA) that if the exact distribution in the restricted direction is chosen, then the result will be the exact solution to the entire problem. Thus, accuracy can be improved in a problem using the Laplace-variational method if the assumed temperature profile can be properly chosen.

In obtaining a high degree of accuracy, however, the expression for the temperature distribution (a double infinite series) in this case becomes so complex that it is difficult to work with. Thus, for the sake of high accuracy the cost of utilization of the temperature distribution will increase considerably.

In considering the economical relationship between accuracy and cost, a comparison of these two factors for two different approximate methods applied to a problem of regular geometry proves interesting. Suppose a certain accuracy is specified and one makes a cost comparison of a finite difference approximate numerical technique to the Laplace-variational technique when both are applied to the two-dimensional problem of Section VA. The development or set-up time for the two solutions cannot be readily compared. Finite difference techniques are already very well formulated and one need only calculate some coefficients and utilize existing computer programs (the digital computer is used in this case)

in an analysis. Since the Laplace-variational technique is new, it is difficult to estimate the amount of time that will be required to arrive at an approximate temperature distribution which has accuracy comparable to a certain finite difference solution for a particular problem. However, the computer utilization necessary for solving the problem by each method can be compared. Suppose one designates the time at which a 100-node temperature distribution is required in the right half of a two-dimensional plate with a step temperature (Section VA). A node spacing of 0.05 ft = 0.6 in. is needed in the finite difference program for a 1 ft x 1 ft plate when 100 nodes are desired. Assume a thermal diffusivity of $0.4 \text{ ft}^2/\text{hr}$. A time step of 0.00125 hr = 4.5 sec is suitable for stability (explicit program). Thus, if one requires the temperature at $t = 0.02 \text{ hrs}$, 16 iterations are needed on the initial temperature distribution. This means approximately 38 seconds in actual computer time is used on a basis of 2.4 sec per iteration. To print out a temperature distribution to match this node spacing and accuracy with the closed form approximation requires about 100 seconds. In this solution a steady state representation of the temperatures is included. Compilation times have been disregarded for both of these programs. Now assume that the temperature distribution is required at $t = 0.10 \text{ hrs}$. The finite difference program requires 80 iterations or 192 seconds of computer time

and for the closed form approximation 100 seconds is again sufficient. At $t = 0.2$ hrs, finite difference solutions require 384 seconds compared to 100 seconds for the closed form approximation.

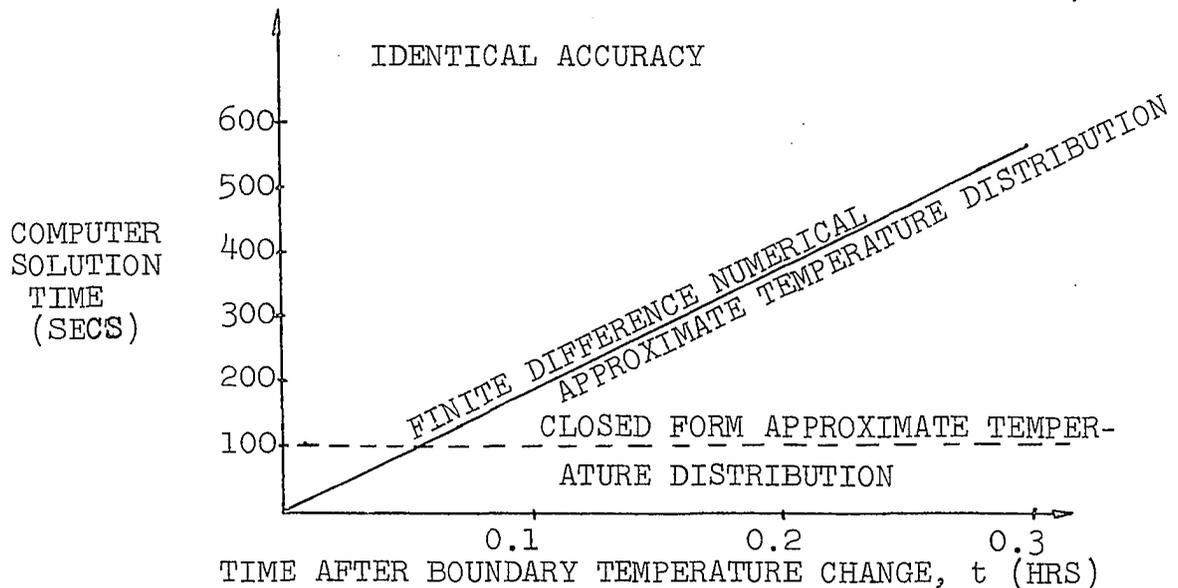


Figure 7. Computer solution time vs time after boundary temperature change

Figure 7 shows the complete comparison between the two methods for all elapsed times under 0.3 hours. The advantages of the approximate analytical solution are evident. They would be even greater for elapsed times greater than 0.3 hours. On the basis of \$150/hr of computing time on the RCA 301S the solution could cost \$24.00 via the approximate numerical approach and \$4.17 via the approximate analytical approach. Although the absolute dollar savings is not of

great magnitude, one can see that the savings could become significant for repeated solutions or at longer elapsed times.

However, another time or cost savings of the closed form approximation will be noticed when a subsequent analysis requiring a mathematical operation on the temperature distribution is needed, e.g., in thermal stress analysis. The point-by-point temperatures of the finite difference routine will either have to be fit to an equation or manipulated in such a way so that a subsequent finite difference analysis can be carried out. The closed form temperature distribution can simply be operated on and it will yield results in a closed form.

Up to this time, the proposed Laplace-variational method has been discussed with respect to penetration type two-dimensional problems only. In such problems the heat seems to penetrate through the body from its surface. When heat generation is involved, however, the heat will build up within the body and then flow to the surface. Such a problem is designated as a non-penetration type. The Laplace-variational method may be used to find approximate temperature distributions in a problem of this type also. In Section VB, just such a non-penetration type problem was considered and Figures 15-17 show the comparison between a 1-term Laplace-variational closed form approximate solution and

the exact temperature distribution. The comparison is favorable at all points in the rectangular plate and improves slightly with time.

Two final considerations must be made to complete this discussion. The first of these has to do with generalizing the Laplace-variational (Ritz or Kantorovich) method to problems in other than two dimensions. Transient heat conduction problems in three dimensions can be solved by the Laplace-variational (Kantorovich) method. However, this is possible only if two of the space dimensions are specified while the remaining dimension is left unrestricted, e.g., $\tilde{T}(x,y,z,s) = (l^2-x^2)(L^2-y^2)Z(z,s)$. The unspecified distribution is then found via variational techniques as before. However, assuming realistic temperature distributions in two dimensions is a difficult task, though the capability of doing so is always present.

At a lower level, a Laplace-variational (Ritz) technique is useful for consideration of transient heat conduction equations in one space dimension. Earlier in this thesis penetration depth problems were considered and were noted to be quite accurate. The difficulty with the penetration depth concept is that in obtaining this accuracy, a non-linearity is introduced into the time variable of the system, i.e., penetration depth = $q_1 = (12\alpha t)^{\frac{1}{2}} \rightarrow q_1^2 = 12\alpha t$. A Laplace-variational (Ritz) combination technique will avoid this

non-linearity; however, the resultant temperature distributions, especially for times close to 0, will sacrifice accuracy. This fact can be observed by following the Laplace-variational development of a one-dimensional example. The same one-dimensional example (Figure 3) as was considered in Section III will now be solved by the Laplace-variational (Ritz) method. This method does not use the penetration and time domain concepts required in the integral and variational formulations. Thus, the approach will vary greatly from that previously utilized in solving this problem.

The general heat conduction equation is given in differential form by Equation 68.

$$\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} = 0 \quad (68)$$

Take the Laplace transformation of this equation with respect to time, with $T(x,0) = 0$, i.e., the initial temperature distribution is 0.

$$\frac{\partial^2 \tilde{T}(x,s)}{\partial x^2} - \frac{s\tilde{T}(x,s)}{\alpha} = 0 \quad (69)$$

where

$$\tilde{T}(x,s) = \int_{t=0}^{t=\infty} T(x,t)e^{-st} dt$$

The Laplace transformed boundary conditions are

$$\frac{\partial \tilde{T}(x=0)}{\partial x} = 0 \text{ and } \tilde{T}(x=l) = \frac{T_0}{s} \quad (70)$$

Assume a quadratic temperature profile of the Ritz form.

$$\tilde{T}(x) = A + Bx + Cx^2 \quad (71)$$

If one substitutes the two boundary conditions into Equation 71, two of the three constants can be determined.

$$\tilde{T}(x) = \frac{T_0}{s} - C(\ell^2 - x^2) \quad (72)$$

The variational form of Equation 69 is given by Equation 73

$$\int_{x=0}^{x=\ell} \left\{ \frac{\partial^2 \tilde{T}(x,s)}{\partial x^2} - \frac{s}{\alpha} \tilde{T}(x,s) \right\} \delta \tilde{T} dx = 0 \quad (73)$$

where the variation in temperature, $\delta \tilde{T}$, is now given by

$$\delta \tilde{T} = \frac{\partial \tilde{T}}{\partial C} \delta C$$

The substitution of Equation 72 into 73 yields Equation 74

$$\int_{x=0}^{x=\ell} (2C - [\frac{T_0}{\alpha} + \frac{sC}{\alpha} (x^2 - \ell^2)]) (x^2 - \ell^2) \delta C dx = 0 \quad (74)$$

If the indicated integration is carried out one obtains the value of the unknown constant, C.

$$C = \frac{T_0}{(2\alpha + \frac{4s\ell^2}{5})} \quad (75)$$

Thus, the Laplace-transformed temperature distribution is known.

$$\tilde{T}(x,s) = \frac{T_0}{s} + \frac{T_0(x^2 - \ell^2)}{(2\alpha + \frac{4s\ell^2}{5})} \quad (76)$$

Simple poles are located at $s = 0$ and $s = -\frac{5\alpha}{2\ell^2}$ and thus the Residue Theorem can be used to find the inversion of $\tilde{T}(x,s)$.

$$\frac{T(x,t)}{T_0} = \left\{ 1 - \frac{5}{4} \left(1 - \frac{x^2}{\ell^2} \right) e^{-\left(\frac{5\alpha}{2\ell^2}\right)t} \right\} \quad (77)$$

$$\text{Let } T^*(x^*, t^*) = \frac{T(x, t)}{T_0}, \quad x^* = \frac{x}{l}, \quad t^* = \frac{\alpha t}{l^2} \quad (78)$$

In order to put Equation 77 in dimensionless form, one must substitute the dimensionless parameters (78).

$$T^*(x^*, t^*) = 1 - 5/4 (1 - x^{*2}) e^{-5/2 t^*} \quad (79)$$

The exact solution to this problem is provided for a comparison (Equation 80).

$$T^*(x^*, t^*) = 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} e^{-\frac{(2n+1)^2 \pi^2}{4} t^*} \left[\cos\left(\frac{(2n+1)\pi}{2} x^*\right) \right] \quad (80)$$

This comparison has been made in Figures 18 and 19. It is evident with reference to these Figures that there is increased accuracy in the temperature distributions as the Fourier Number (dimensionless time) increases. The lack of such accuracy at small Fourier Numbers is brought out distinctly in Figure 19.

The final point, which must be considered, is concerned with the limitations of the Laplace-variational method. It is extremely difficult to apply a Laplace transformation to a non-linear heat conduction problem. For this reason, the limitation that the problem be one of linear heat conduction has been placed on the method. A second limitation is concerned with the type of body in which the temperature distribution is desired. If the body is irregularly shaped then it will probably be difficult to apply this method in

its analysis. Assuming approximate temperature profiles that will properly describe the temperature distribution in the restricted direction will be extremely difficult. Thus, the limitation of the Laplace-variational method will also be regulated by the shape of the body considered.

VII. SUMMARY AND CONCLUSIONS

Integral and variational methods are analytical techniques which provide closed form approximate solutions to one-dimensional heat conduction problems. However, because of the concepts involved it is difficult to apply these methods in cases of two-dimensional heat flow. Therefore, an analytical method has been developed which will provide closed form approximate temperature distributions for two-dimensional transient conduction heat transfer problems. It will be referred to as a Laplace-variational method since it utilizes a Laplace transformation along with methods from the calculus of variations.

The conclusions that can be reached concerning the method are

- 1) The Laplace-variational method can be applied to bodies which are subject to a penetration type boundary condition or to bodies with heat generation. Application of the method to both of these types of two-dimensional problems is shown in this dissertation.
- 2) Some one-dimensional and three-dimensional transient conduction heat transfer problems can be analyzed by the Laplace-variational method even though the method was formally prepared for two-dimensional problems. The procedure for finding

temperature distributions for one-dimensional problems by this method is given in the "Discussion of the Results" along with an example of its usage. In general, the Laplace-variational method will be more advantageous than other approximate analytical one-dimensional methods all of which introduce complexities into the solution. The procedure for finding three-dimensional temperature distributions in conduction heat transfer problems via the Laplace-variational approach is also mentioned in the "Discussion of the Results".

- 3) Accuracy will normally be sacrificed somewhat when the Laplace-variational method is utilized since an approximate solution results from the analysis. Of course, the accuracy of a temperature distribution found by this method can be improved if one so desires. The two-dimensional penetration type problem analyzed in this paper shows that even an exact solution can be found via the Laplace-variational method if accuracy is required. However, when too high a degree of accuracy is necessary, much of the savings gained by the approximate solution is lost. Economy requires a proper relationship between accuracy and cost.
- 4) Use of the Laplace-variational method in heat transfer problems means that the thermal properties

of a material must be constant and also that a radiation boundary condition cannot be considered. Utilization of the technique in this thesis has also been restricted to problems involving regular geometry. The applicability of the method is not confined to regular geometry but it is for this type of problem that the method will have its greatest economic usefulness. In most instances it will be very difficult to assume the form of the temperature profiles required as an assumption in beginning the problem solution for irregular geometries.

- 5) On the basis of computer time required, a comparison was made between the approximate numerical and approximate analytical techniques when obtaining identical accuracies for a regular-shaped two-dimensional heat conduction problem. Figure 7 shows the advantages in computer time requirements afforded by the approximate analytical method for all time periods except those extremely near the initiation of boundary condition change. At times very near the time of boundary condition change the approximate numerical method is superior.

VIII. RECOMMENDATIONS FOR FURTHER STUDY

In the course of this thesis a method has been developed which enables one to find approximate closed form temperature distributions in two-dimensional transient conduction heat transfer problems. However, even though the general concepts of the method are now apparent much can still be done to expand on the applicability of method. Some recommendations for further study are listed below.

- 1) Many repetitious problems occur in heat conduction analysis and in particular in reactor vessel analysis. For this reason, a study which would classify the assumed Laplace-variational profiles for certain specific geometrical configurations and each type of boundary condition would be worthwhile. Then, these cases could be readily considered when they arise.
- 2) The procedure for utilizing the Laplace-variational method to determine temperature distributions in a three-dimensional transient conduction heat transfer problem is mentioned in this dissertation. Though the application of the method to three-dimensional problems is thus displayed as conceptually possible, the feasibility of applying it economically has not yet been shown. The types of assumed profiles required to provide relatively accurate temperature

distributions would need to be considered and classified. It is recommended that the extension of the applicability of the Laplace-variational method to three-dimensional problems be undertaken.

- 3) An investigation of Piot's variational method (References 25-35) for the purpose of extending the capability of finding simple closed form solutions to nonlinear two-dimensional transient conduction heat transfer problems should be carried out. This would enable one to consider variable thermal properties of materials as well as the radiation boundary condition.

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X. NOMENCLATURE

<u>Symbol</u>	<u>Term</u>	<u>Units</u>
A	Surface of system	sq ft
B	General symbol for a mathematical operator	
c	Specific heat of a body	Btu/lb-°F
k	Thermal conductivity of a body	Btu/hr-ft-°F
L	Length of a body	ft
ℓ	Thickness of a body	ft
q	Heat flux	Btu/hr-sq ft
q'''	Internal heat generation	Btu/hr-cu ft
q ₁ (t)	Penetration depth representing the distance into a body beyond which there is negligible heat penetration (a function of time)	ft
s	Laplace transformation parameter	1/hr
t	Time	hr
t*	Fourier Modulus = $\frac{\alpha t}{\ell^2}$ = Dimensionless time	
t _{q₁}	Penetration time, i.e., time for heat to penetrate through a plate	hr
T	Temperature	°F
T*	Dimensionless temperature term = $\frac{T}{T_0}$ or $\frac{Tk}{q''' \ell^2}$	
T ₀	Step change in temperature	°F
\tilde{T}	Laplace transformed temperature	°F-hr
u	Velocity	Ft/hr
u _s	Stream velocity	Ft/hr

x	Space coordinate	ft
x^*	Dimensionless space coordinate = $\frac{x}{l}$	
$X(x,s)$	Assumed space distribution in x	hr- ^o F/sq ft
$\delta X(x,s)$	Variation in X	hr- ^o F/sq ft
y	Space coordinate	ft
y^*	Dimensionless space coordinate = $\frac{y}{l}$	
$Y(y,s)$	Assumed space distribution in y	hr- ^o F/sq ft
$\delta Y(y,s)$	Variation in Y	hr- ^o F/sq ft
$\delta T(x,y,s)$	Variation in T	hr- ^o F
α	Thermal diffusivity of body = $\frac{k}{\rho c}$	sq ft/hr
ρ	Density of body	lb/cu ft
δ	Laminar boundary layer	ft

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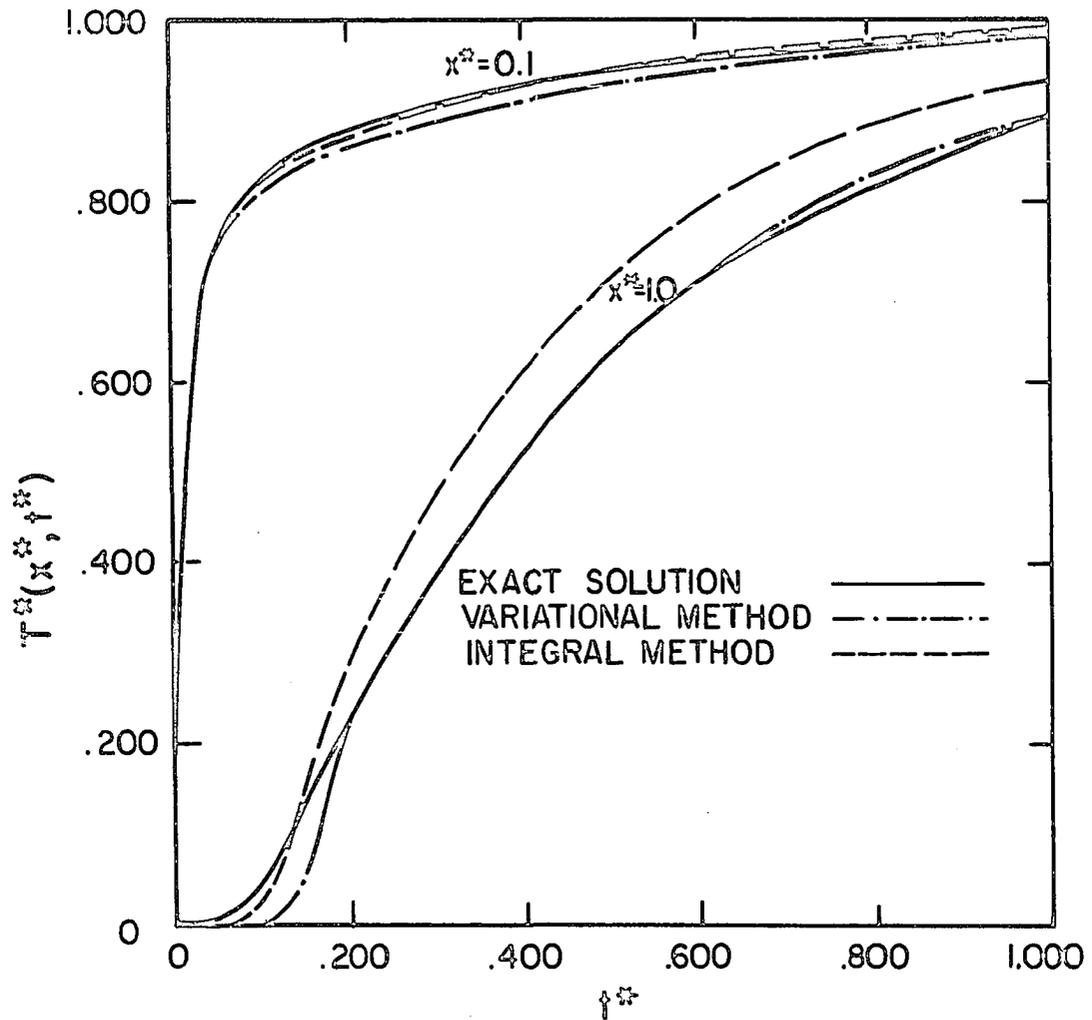
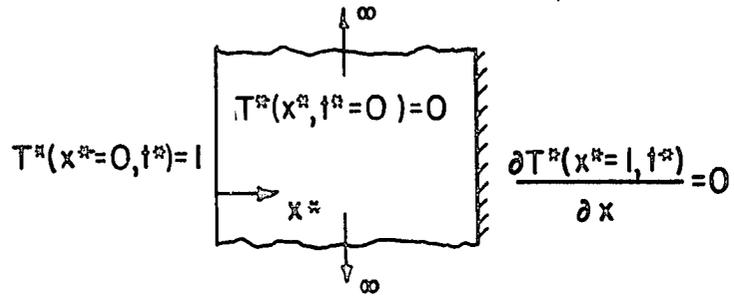


Figure 8. Temperature as a function of Fourier modulus in a 1-D plate with a step temperature

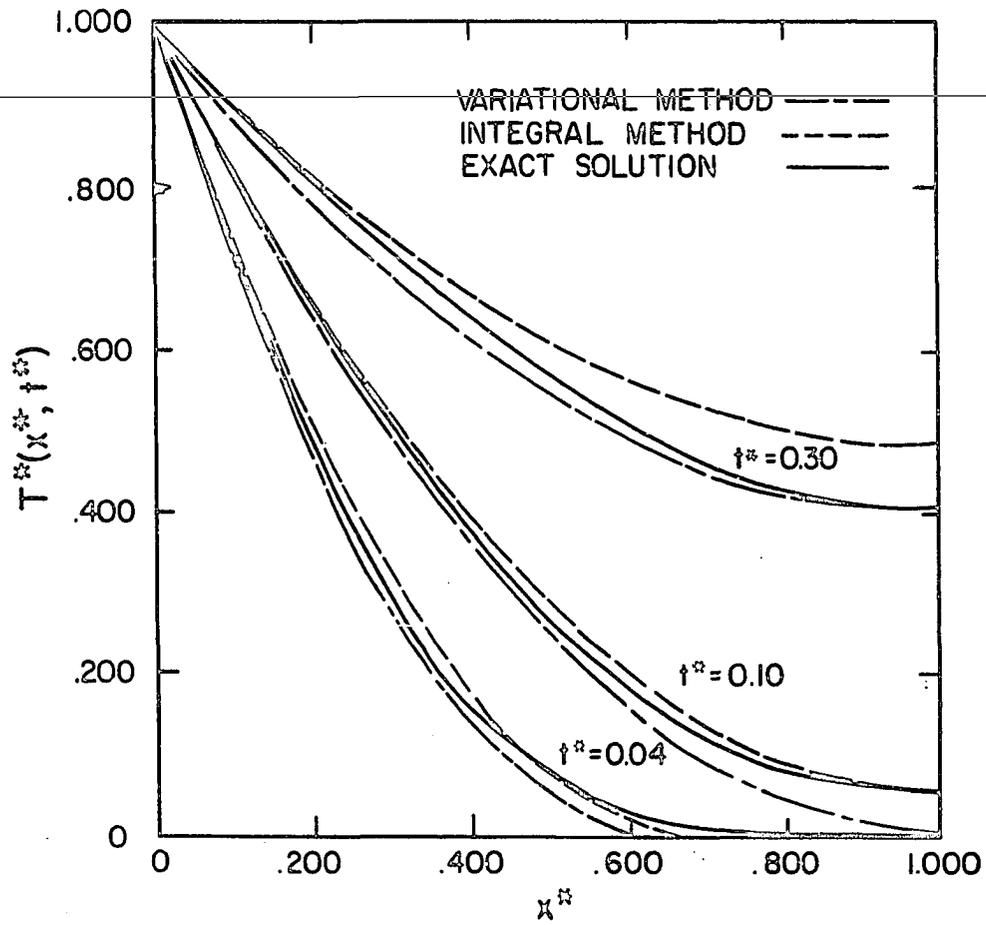
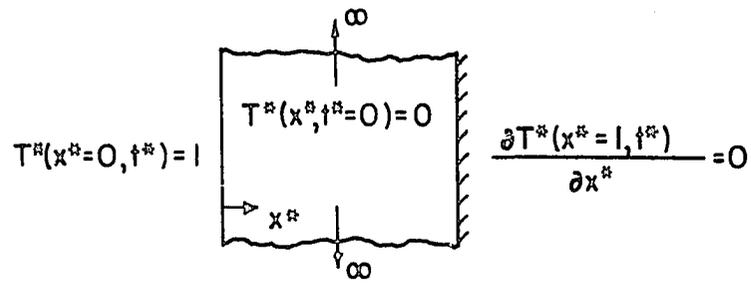


Figure 9. Temperature as a function of distance in a 1-D plate with a step temperature

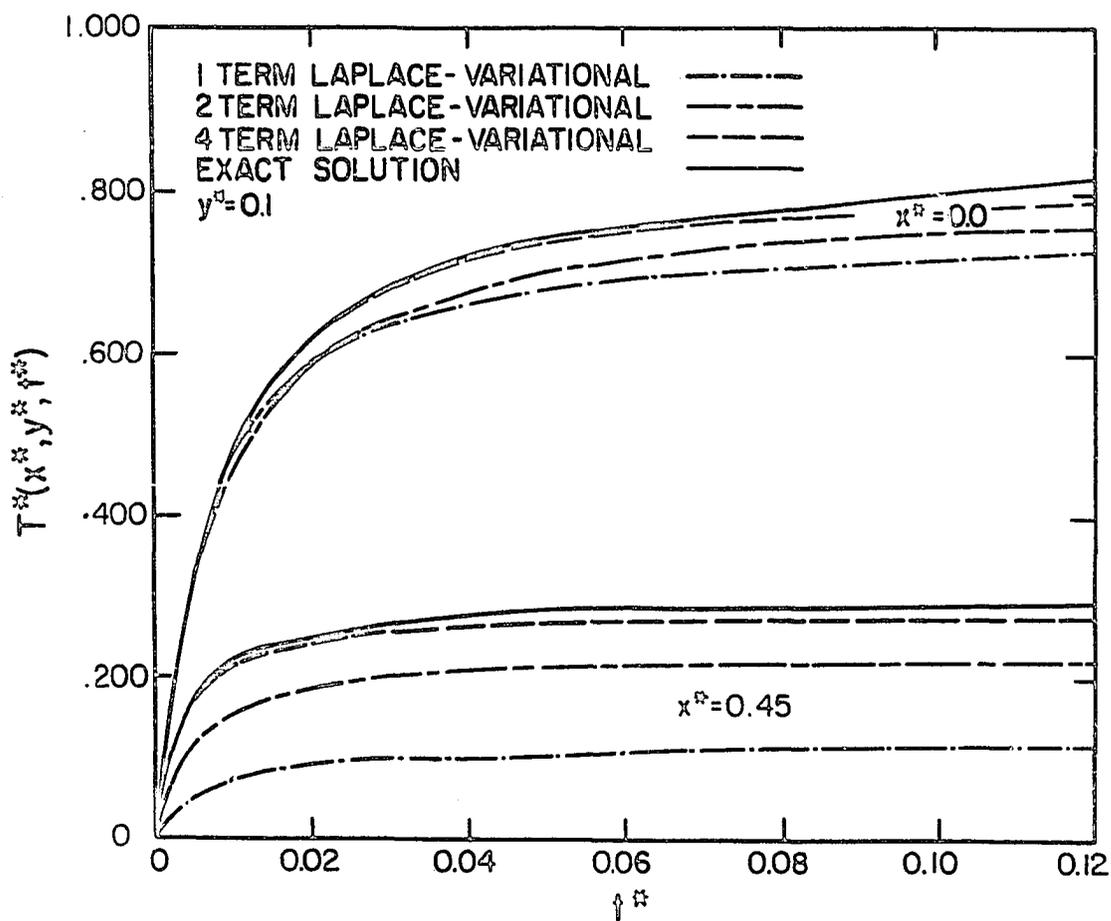
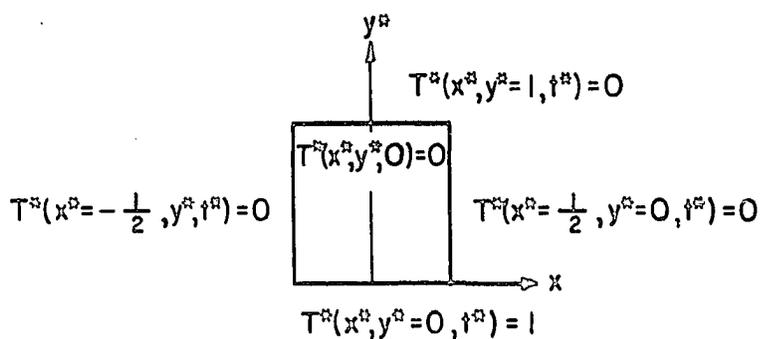


Figure 10. Temperature as a function of Fourier modulus in a 2-D plate with a step temperature

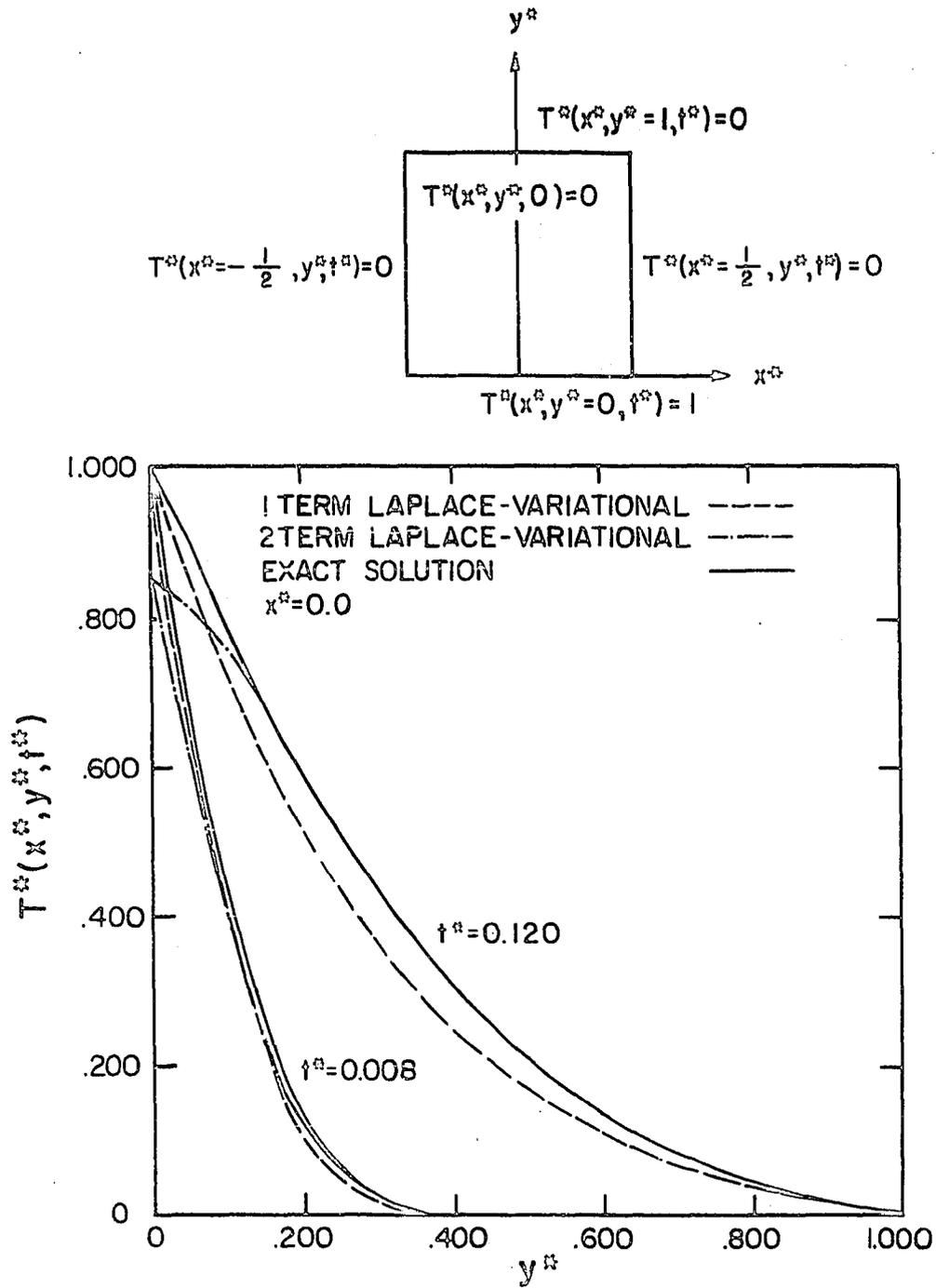


Figure 11. Temperature as a function of y^* distance in a 2-D plate with a step temperature

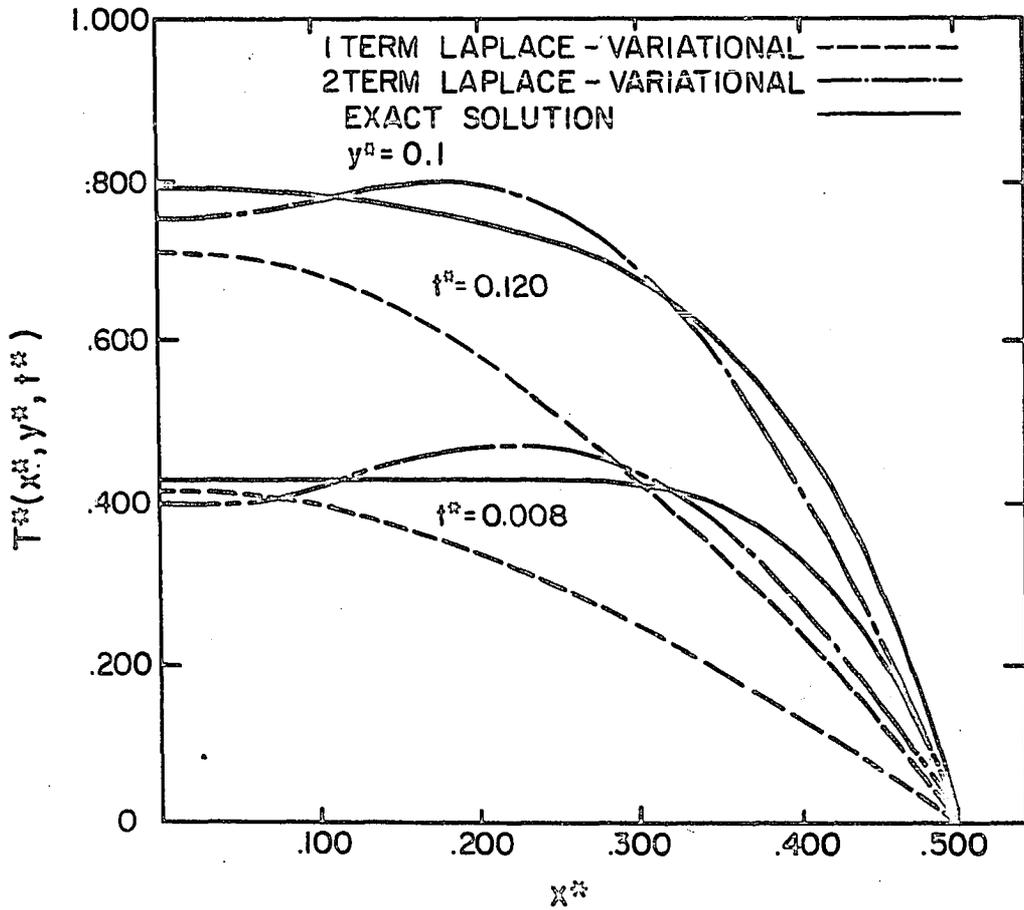
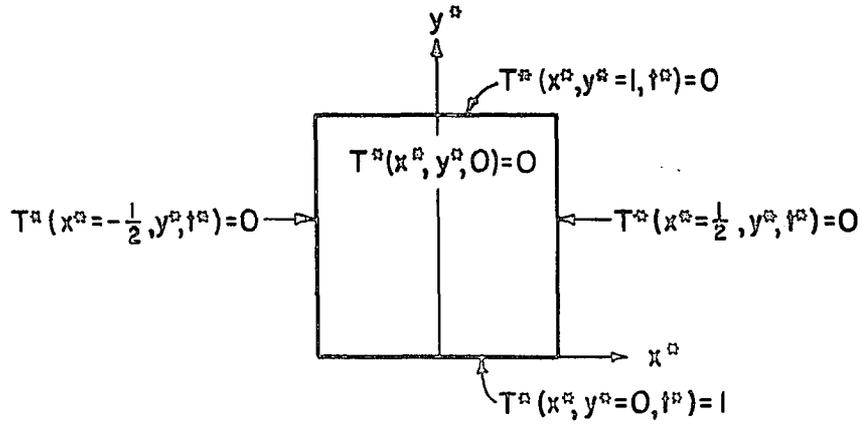


Figure 12. Temperature as a function of x^* distance in a 2-D plate with a step temperature

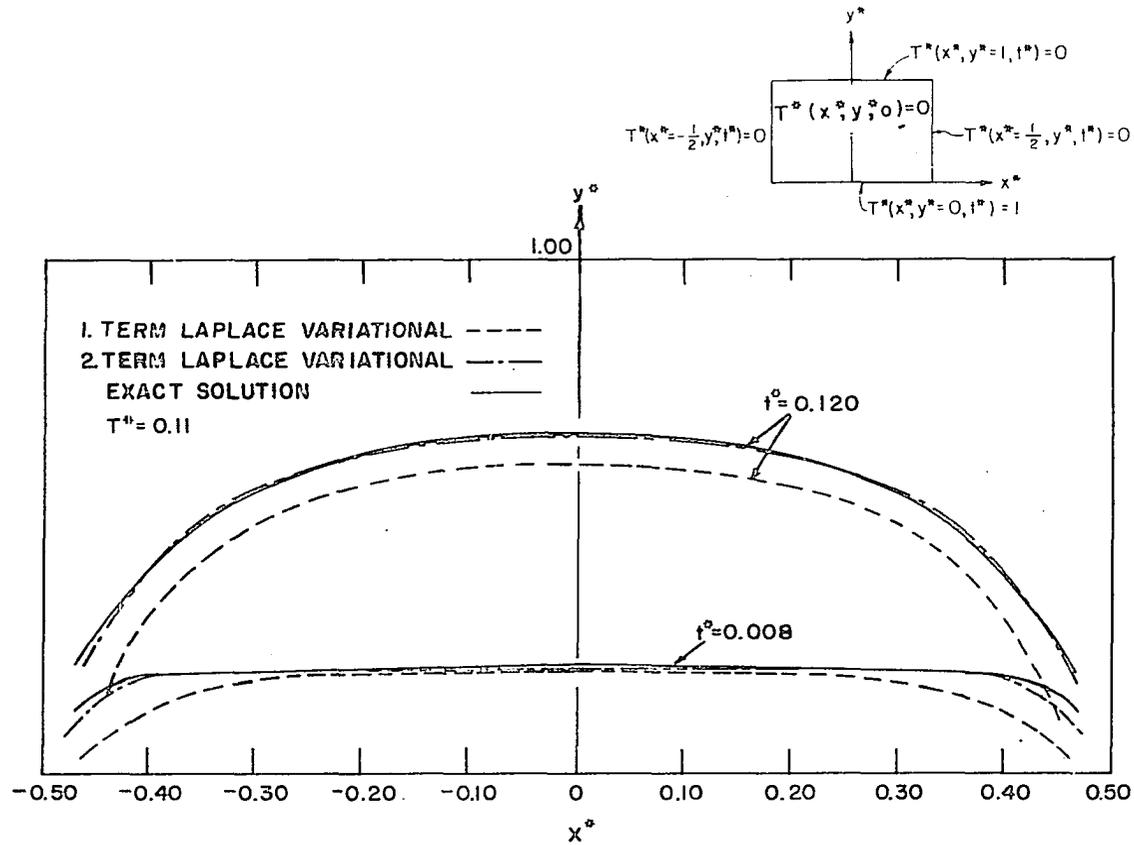


Figure 13. Contour map of the isotherm $T^* = 0.11$ at Fourier numbers of $t^* = 0.008$ and $t^* = 0.120$

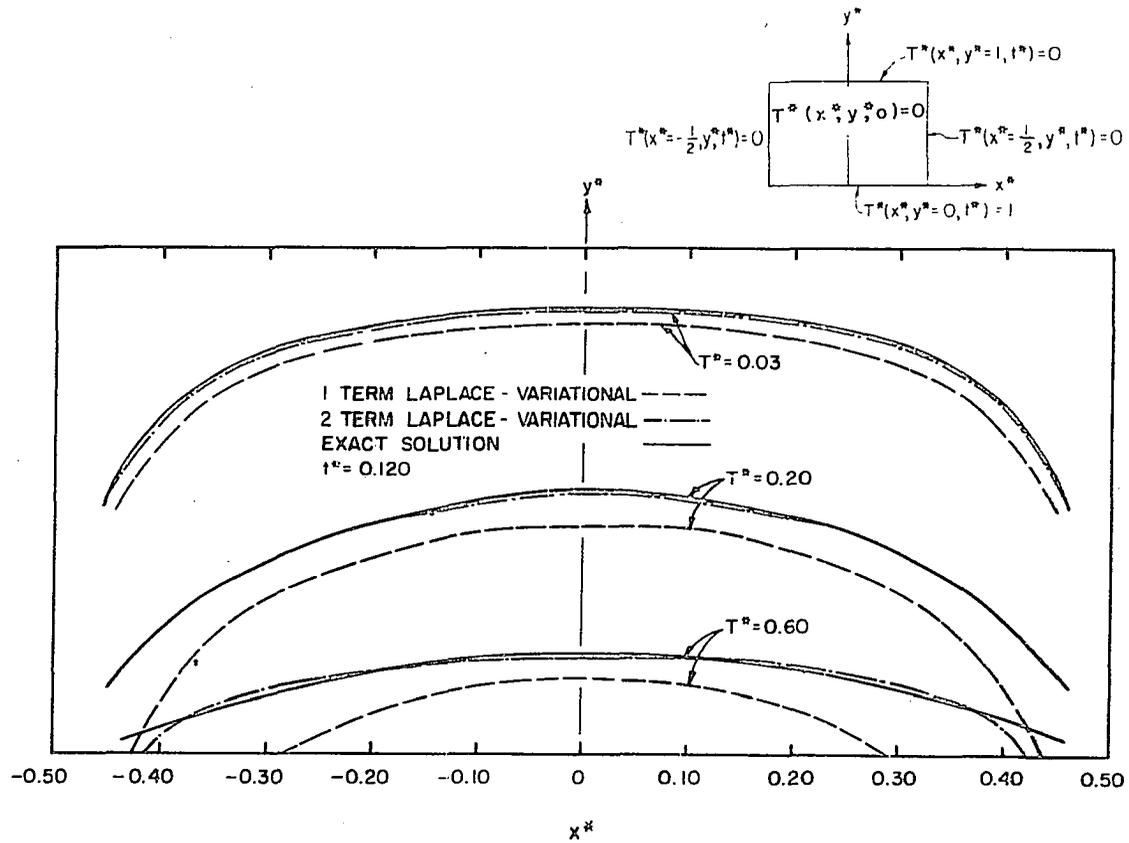


Figure 14. Contour map of the isotherms $T^* = 0.03, 0.20$ and 0.60 at a Fourier number of $t^* = 0.120$

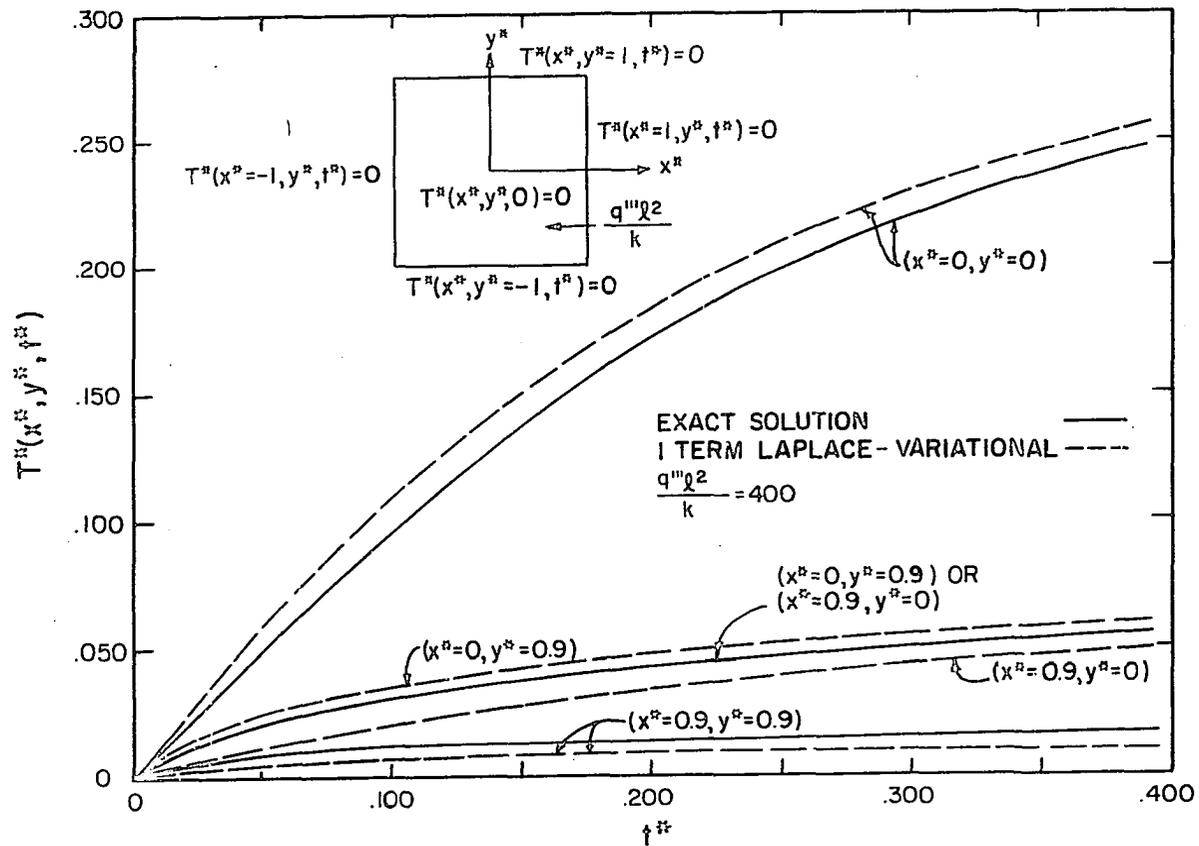


Figure 15. Temperature as a function of Fourier modulus in a 2-D plate with heat generation.

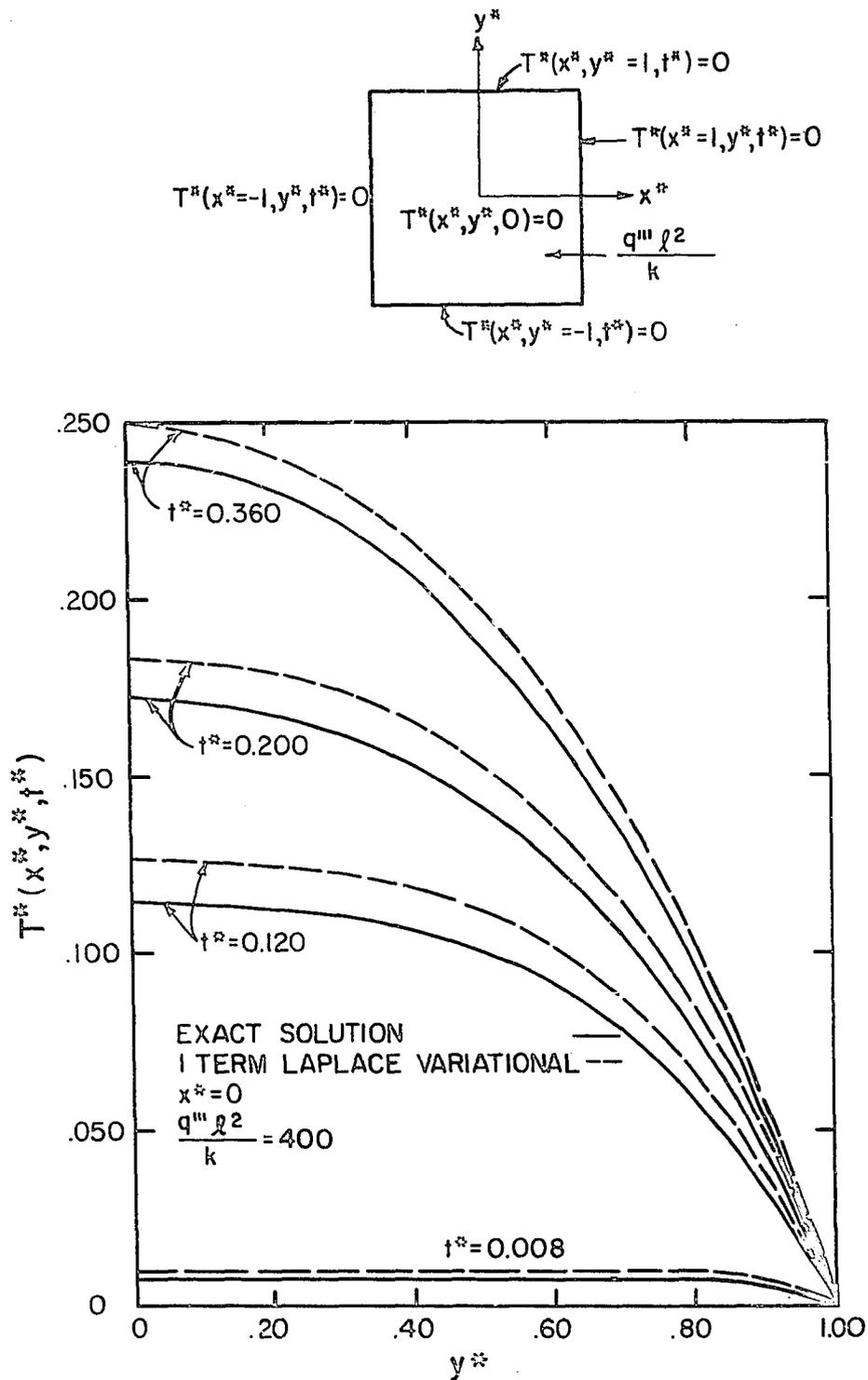


Figure 16. Temperature as a function of y^* location for $x^* = 0$ at different Fourier numbers in a 2-D plate with heat generation

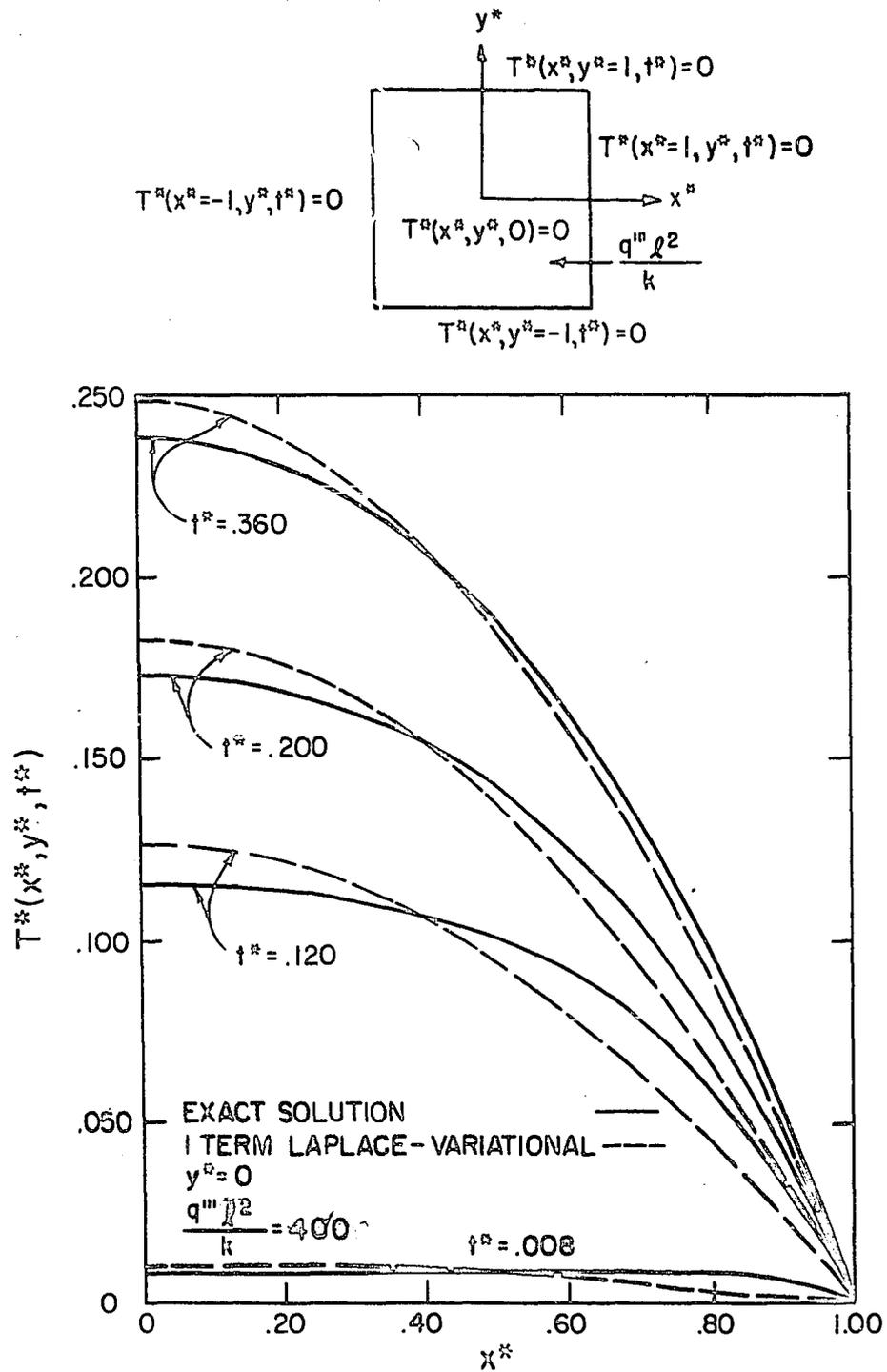


Figure 17. Temperature as a function of x^* location for $y^* = 0$ at different Fourier numbers in a 2-D plate with heat generation

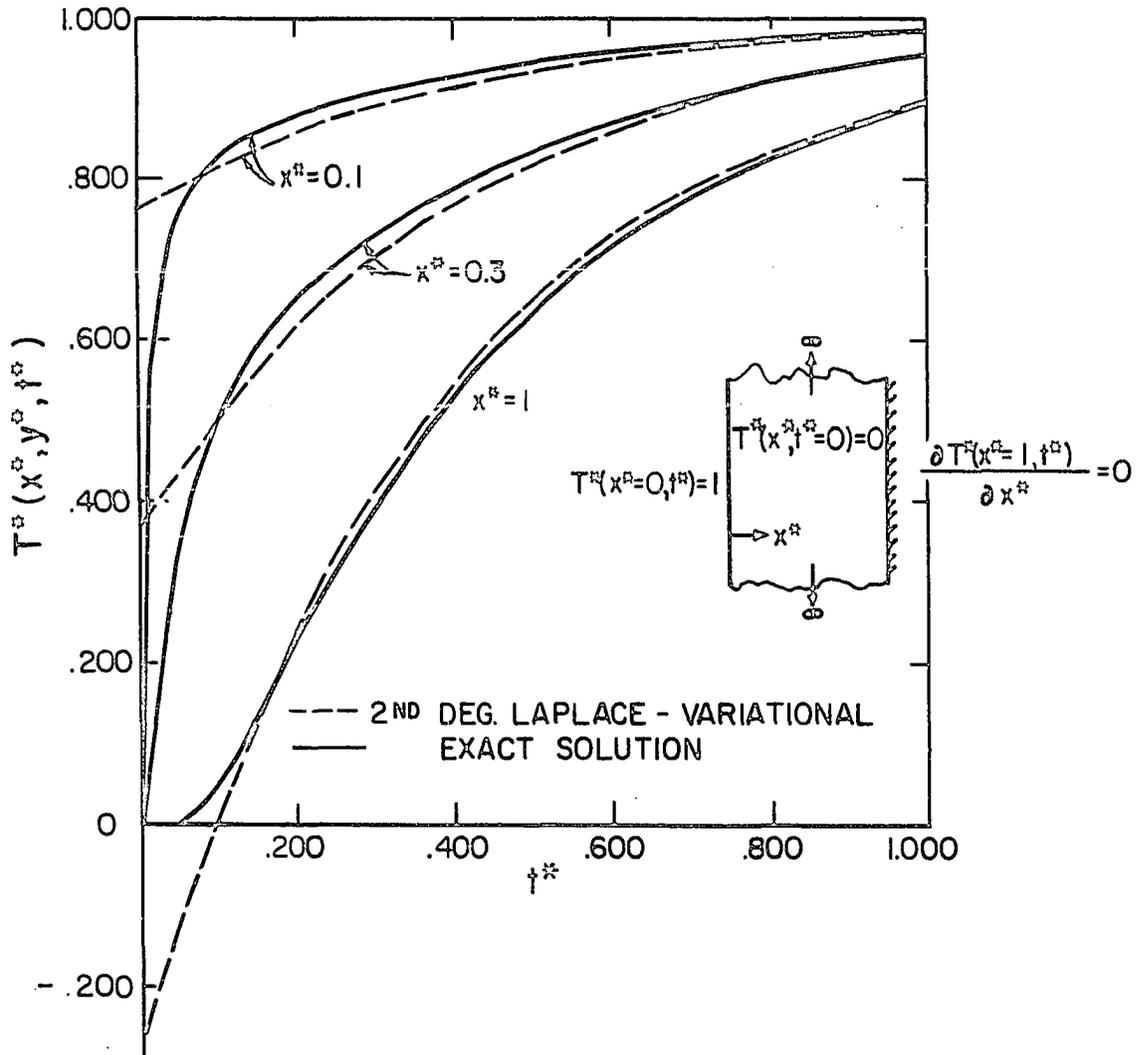


Figure 18. Temperature as a function of Fourier modulus in a 1-D plate with a step temperature

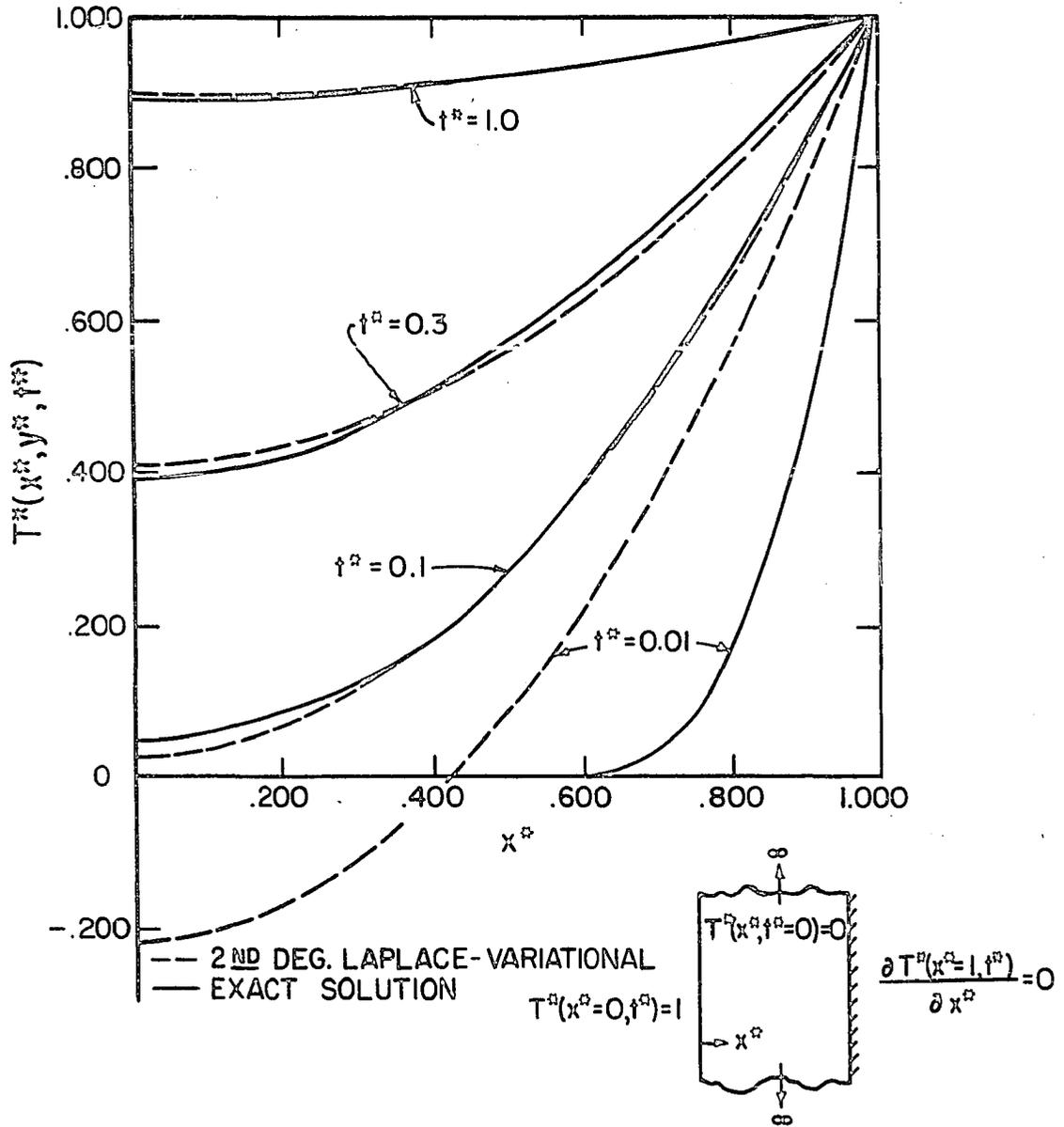


Figure 19. Temperature as a function of x^* distance in a 1-D plate with a step temperature